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The pre-history of Kenneth Arrow's social choice and individual values

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Abstract The purpose of this article is to give an historical sense of the intellectual developments that determined the form and content of Kenneth Arrow's path-breaking work published in 1951. One aspect deals with personal influences that helped shape Arrow's own thinking. A second aspect is concerned with the early history of the general theory of relations, which is mainly centered in the nineteenth century, and also with the essentially independent modern development of the axiomatic method in the same time period. Arrow's use of general binary relations and of axiomatic methods to ground, in a clear mathematical way, his impossibility theorem marks a turning point in welfare economics, and, more generally, in mathematical economics.

1 Some foundational remarks on the history of ordinal utility

We can begin with a well-known quotation from Hicks and Allen (1934).

> Of all Pareto's contributions there is probably none that exceeds in importance his demonstration of the immeasurability of utility. To most earlier writers, to Marshall, to Walras, to Edgeworth, utility had been a quantity theoretically measurable; that is to say, a quantity which would be measurable if we had enough facts. Pareto definitely abandoned this, and replaced the concept of utility by the concept of a scale of preferences. It is not always observed that this change in concepts was not merely a change of view, a pure matter of methodology; it rested on a positive demonstration that the facts of observable conduct make a scale of preferences capable of theoretical construction (in the same sense as above) but they do not enable us to proceed from the scale of preference to a particular utitlity function.
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> Hicks and Allen 1934, p. 52

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As the quotation makes clear, the originator of the ordinal theory of utility is Pareto, particularly in view of his claim to demonstrate the inability to measure utility on anything like a cardinal scale. But Hicks and Allen did not introduce any radical change, nor did Pareto, from thinking of utility, cardinal or ordinal, as being represented by continuous functions of quantities of goods, ideally continuous, also. A rigorous proof of the existence of such continuous functions under a general topological condition of separability was given by Debreu (1954).

The explanation of the shift away from cardinal measures of utility is given a more psychological and behavioristic account in the following quotation from Samuelsons well-known book, *Foundation of Economic Analysis* (1948, p. 91).

> ...there has been a shift in emphasis away from the physiological hedonistic, introspective aspects of utility. Originally great importance was attached to the ability of goods to fill basic biological needs; but in almost every case this view has undergone extreme modification. At the same time, there has been a similar movement away from the concept of utility as a sensation, as an introspective magnitude. It is not merely that the modern economist replaces experienced sensation or satisfaction with anticipated sensation, desire, according to the now familiar distinction between ex post and ex ante analysis. But much more than this, many writers have ceased to believe in the existence of any introspective magnitude or quantity of a cardinal, numerical kind. With this skepticism has come the recognition that a cardinal measure of utility is in any case unnecessary; that only an ordinal preference, involving 'more' or 'less' but not 'how-much,' is required for the analysis of consumer's behavior.
> Samuelson 1948, p. 91

Three years later, with the appearance of Arrow's *Social Choice and Individual Values* (1951), a new turn was taken, as is well stated by Arrow himself in the opening pages. (A retrospective personal account by Ken on his discovery of the Impossibility Theorem is given in Arrow (1991).)

> In this study it is found convenient to represent preference by a notation not customarily employed in economics, though familiar in mathematics and particularly in symbolic logic. We assume that there is a basic set of alternatives which could conceivably be presented to the chooser. In the theory of consumers choice, each alternative would be a commodity bundle; in the theory of the firm, each alternative would be a complete decision on all inputs and outputs; in welfare economics, each alternative would be a distribution of commodities and labor requirements. In general, an alternative is a vector; however, in the theory of elections, the alternatives are candidates. These alternatives are mutually exclusive; they are denoted by the small letters x, y, z,.... On any given occasion, the chooser has available to him a subset S of all possible alternatives, and he is required to choose one out of this set.
> Arrow 1951, pp. 11–12

This shift from the tradition inspired by Pareto and continuing through Samuelson, along with many other economists, was a deliberate, not an accidental one, on Arrow's part. The main purpose of this article is to try to sketch some of that prehistory of this move of Arrow's and in what way can something be said about his own personal background in making it. Ken has told me that already in high school in the 1930s he read and was influenced by Bertrand Russell's *Introduction to Mathematical Philosophy* (1920).

An important remark, providing a very good additional clue, is to be found in Alfred Tarski's *Introduction to Logic*, first published in 1941. In the preface, dated 1940, he says the following.

> I also owe many thanks to Mr. K. J. Arrow for his help in reading proofs.
> Tarski 1941a, p. xvi

The fact is that Ken was an undergraduate student at City College of New York and he took at this time a course in the calculus of relations from Tarski, who was temporarily teaching there. (Ken graduated from CCNY in 1940.) To give a sense of the sort of attitude that Tarski held, let me quote from Chapter 5 of the logic text mentioned. This is Tarski on the general theory of relations, but stated at a very elementary level.

> We will now discuss several concepts belonging to the general THEORY OF RELATIONS, which constitutes a special and very important part of logic, and in which relations of an entirely arbitrary character are considered and general laws concerning them are established.
> Tarski 1941a, pp. 87–88

Ken refers to Tarski's logic text in a footnote on p. 13, and one also on p. 14 concerning terminology in the theory of relations (Arrow 1951, pp 13–14). Moreover, in all likehood the course Ken attended in 1940 probably included most of the material published in Tarski (1941b) on the calculus of relations. But the standard operations on relations covered in the article, such as converse and relative product, were not used by Ken, and subsequently have played only a minor role in the history of formal theories of choice. (Ken has informed me that he did later use the calculus of relations, in particular, relative powers, which are defined recursively in terms of relative products, in Arrow (1976).

It was the clear abstract formulation at an elementary level of ordering relations that had the largest impact. Ken has said to me he probably first learned about ordering relations from a course in algebra using Birkhoff and MacLanes well-known text (1941), as well as from Tarski.

2 Theory of relations from De Morgan to Tarski

The formal theory of relations began with De Morgan. To get Tarski's own views on the history of the general theory of relations, here is a long quotation from his article on the calculus of relations.

> The logical theory which is called the *calculus of (binary) relations*, and which will constitute the subject of this paper, has had a strange and rather capricious line of historical development. Although some scattered remarks regarding the concept of relations are to be found already in the writings of medieval logicians, it is only within the last hundred years that this topic has become the subject of systematic investigation. The first beginnings of the contemporary theory of relations are to be found in the writings of A. De Morgan, who carried out extensive investigations in this domain in the fifties

> of the nineteenth century. De Morgan clearly realized the inadequacy of traditional logic for the expressions and justification, not merely of the more intricate arguments of mathematics and the sciences, but even of simple arguments occurring in everyday life; witness his famous aphorism, that all the logic of Aristotle does not permit us, from the fact that a horse is an animal, to conclude that the head of a horse is the head of an animal. In his effort to break the bonds of traditional logic and to expand the limits of logical inquiry, he directed his attention to the general concept of relations and fully recognized its significance. Nevertheless, De Morgan cannot be regarded as the creator of the modern theory of relations, since he did not possess an adequate apparatus for treating the subject in which he was interested, and was apparently unable to create such an apparatus. His investigations on relations show a lack of clarity and rigor which perhaps accounts for the neglect into which they fell in the following years.
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> The title of creator of the theory of relations was reserved for C.S. Peirce. In several papers published between 1870 and 1882, he introduced and made precise all the fundamental concepts of the theory of relations and formulated and established its fundamental laws....
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> ...Pierce's work was continued and extended in a very thorough and systematic way by E. Schröder. The latter's *Algebra und Logik der Relative*, which appeared in 1895 as the third volume of his *Vorlesungen über die Algebra der Logik*, is so far the only exhaustive account of the calculus of relations.
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> Tarski 1941b, pp. 73–74

The author that Tarski and others recognized as the modern source of general theory of relations, Augustus De Morgan, was a professor at the University of London in the mid-nineteenth century. It is sometimes claimed that Leibniz anticipated De Morgan in several ways. If so, it is hard to say precisely how. For a definitive account of the many fragmentary texts, and a detailed sense of the difficulty of a unified interpretation, see Mates (1986, Ch. XII).

De Morgan's literary style is much better than his rather tortuous formal notation that I shall not reproduce. Here is a series of quotations from one of his most extensive works.

> Much has been written on relation in all its psychological aspects except the logical one, that is, the analysis of necessary laws of thought connected with the notion of relation. The logician has hitherto carefully excluded from his science the study of relation in general: he places it among those heterogeneous *categories* which turn the porch of his temple into a magazine of raw material mixed with refuse... Terms may be related, even though they have more meaning than just goes to the relation.
>
> De Morgan 1860/1966, p. 208

The next one is an amusing footnote.

> The old riddle-books often profound the following query:–If Dick's father be Tom's son, what relation is Dick to Tom? When a boy, I heard the following classical and Protestant version of the puzzle, over which I have since made grown persons ponder, not always with success. An abbess observed that an elderly nun was often visited by a young gentleman, and asked what relation he was. "A very near relation," answered the nun; "his mother was my

> mother's only child:" which answer, as was intended, satisfied the abbess that the visitor must be within the unprohibited degrees, without giving precise information.
>
> De Morgan 1860/1966, p. 213fn

The next quotation shows how far the austere contemporary set-theoretical language of relations used by Ken is from the speculative beginnings.

> Any two objects thought brought together by the mind, and thought together in one act of thought, are in *relation*. Should any one deny this by producing two notions of which he defies me to state the relation, I tell him that he has stated it himself: he has made me think the notions in the relation of *alleged impossibility of relation*; and has made his own objection commit suicide. Two thoughts cannot be brought together in thought except by a thought: which last thought contains their *relation*.
>
> De Morgan 1860/1966, p. 218

The same line of thought is found in the closing remark of De Morgan's.

> I expect agreement in what I have said neither from the logicians nor from the algebraists: but, for reasons given in my last paper, I do not submit myself to either class. Not that I by any means take it for granted that all those who have cultivated both sciences will agree with me. When two countries are first brought by the navigators into communication with each other, it is found that there are two kinds of perfect agreement, and one case of nothing but discordance. All the inhabitants of each of the countries are quite at one in believing a huge heap of mythical notions about the other. At first, the only persons who though similarly circumstanced nevertheless tell different stories are the very mariners who have passed from one land to the other.
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> De Morgan 1860/1966, pp. 241–242

As already remarked, what I avoided is reproducing and explaining De Morgans chaotic and antiquated notation. As the passages I have quoted show, he was conceptually breaking a new ground, much of it clearly for the first time, as when he says "A relation is transitive when a relative of a relative is a relative of the same kind." (1860/1966, p. 226). This is equivalent to the modern definition in the calculus of relations: A binary relation R is transitive if and only if $R/R \subseteq R$, where / is the relative product and $\subseteq$ is set inclusion. In fact, in various places De Morgan has a rather thorough discussion of the properties of transitive relations, e.g., if R is transitive, its converse is also. But he does not go on to examine types of ordering relations.

I now turn to Charles Saunders Peirce. Following De Morgan's groundbreaking article in the Cambridge Philosophical Transactions in 1860, the next fundamental contribution was Peirce's long and systematic article published in 1870, which begins by acknowledging De Morgan's work, and then point out some of the deficiencies of De Morgan's concepts and notation. There is, unfortunately, a plethora of clear and well-defined notation, now mostly obsolete, in Peirce's work.

It is, moreover, pretty obvious why it is not pertinent in the present paper to delve more carefully into the detailed notation created by De Morgan and Peirce. This notation is almost entirely focused on the calculus of relations, i.e., operations on relations, not on the use of relations, binary ones especially, to develop various kinds of ordering concepts. So, for example, Ken discusses at various points in his 1951 book, strong orderings, weak orderings, partial orderings, quasi-orderings and social orderings, but never relative products of relations, the converse of relation, or the image of a set under a relation (the standard generalization of a functional mapping). And he is not alone in this, but typical of authors working in social-choice theory, as might be expected from the seminal character of his 1951 publication.

Peirce does formulate some of the elementary ordering relations listed above, for example in Peirce (1870, Section 2), he states essentially the axioms for partial orderings (reflexive, antisymmetric and transitive), and is probably the first person to do so explicitly, at least so Birkhoff (1948) claims. A little later (1881), Peirce explicitly defines a *system of quantities* as a partial ordering–but he does not use this terminology. If the relation is strongly connected as well, it is a *simple* system of quantities. I believe it is correct to say that Peirce nowhere has a formulation of general or abstract ordering relations.

After Peirce, the next major figure in developing the theory of relations was Schröder (1890–1995), who thoroughly explored the subject in three volumes, including partial orderings, but I will not attempt any survey here. The quotation from Tarski clearly states Schröder's place in the systematic development of the calculus of relations.

3 Arrow and the axiomatic method in economics

One curious point about the history of the theory of relations in the nineteenth century concerns its connection to the foundations of geometry. Schröder's massive work appeared in the same decade as Hilbert's *Foundation of Geometry* (1897), the work that was the culmination of several decades of providing a rigorous axiomatic foundation for Euclidean geometry. In many ways, Pasch's explicit analysis of the axiomatic method in geometry (1882), published 15 years earlier, was the fundamental methodological step of formalization. Here is a critical passage (the translation is taken from an excellent historical account by Nagel 1939, of the foundations of geometry in the nineteenth century).

> Indeed, if geometry is to be really deductive, the deduction must everywhere be independent of the *meaning* of geometric concepts, just as it must be independent of the diagrams; only the *relations* specified in the propositions and definitions employed may legitimately be taken into account. During the deduction it is useful and legitimate, but in *no* way necessary, to think of the meanings of the terms; in fact, if it is necessary to do so, the inadequacy of the proof is made manifest. If, however, a theorem is rigorously derived from a set of propositions — the *basic* set — the deduction has a value which goes beyond its original purpose. For if, on replacing the geometric terms in the basic set of propositions by certain other terms, true propositions are obtained, then corresponding replacements may be made in the theorem; in this way we obtain new theorems as consequences of the altered basic propositions without having to repeat the proof.
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> Pasch 1882, pp. 237–238

But there is a little connection here. Neither De Morgan nor Peirce thought geometry was of much help in the theory of relations. And, to my knowledge, neither of them were mentioned by Pasch or Hilbert in the nineteenth century, but only later, by Hilbert in his "logic" phase, e.g., in the Introduction to Hilbert and Ackerman (1928/1950, p. 1).

Ken's axiomatization follows exactly Pasch's recipe for geometry, which, of course, is the subject whose axiomatization led the way for other parts of mathematics from ancient Greek days until the general spread of the modern viewpoint in mathematics in the twentieth century.

This brings us to a second, closely related point. From a pedagogical standpoint, no clearer and more elegant exposition of the axiomatic method is to be found in the first half of the twentieth century than that given in Chapter VI of Tarski's logic text (1941), which Ken proofread and undoubtedly understood well. The axiomatic method described in general elementary form by Tarski was already familiar enough in pure mathematics but not in economics. The theory of welfare economics embodied in the publications prior to Ken's work (1951) were substantive, and sometimes mathematical, when written by economists such as Samuelson or Lange, but not in a rigorously axiomatic way. Ken also led an irreversible change in introducing such axiomatic methods into welfare economics. I am not suggesting that Ken learned about the axiomatic method only from Tarski. The pure mathematics of other sorts he was learning, both as an undergraduate at CCNY and as graduate student at Columbia. He has told me that the biggest influence on him at Columbia was the distinguished mathematical statistician Harold Hotelling, who was in fact in the Department of Economics. But Hotellings course in mathematical economics was not at all oriented toward the axiomatic approach.

The important point, not lost on any student of economic methodology, is the stylistic change from Samuelson's *Foundation of Economic Analysis* (1948), a shining example of the robust and effective use of mathematics in economics, written like a superb treatise of theoretical physics, to Ken Arrow's *Social Choice and Individual Values* (1951), published only three years later. Ken's work is written in the modern axiomatic style of pure mathematics, now dominant in large parts of mathematical economics.

Acknowledgements A number of remarks in this article are based on conversation with Ken Arrow. I acknowledge with pleasure his willingness to think about his own earlier years and thereby to both clarify and amplify many of the points I have tried to make. Of course, the reader needs to realize that, on some matters, I may not have understood in the most accurate way his remarks to me.

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