The Ability of Elementary-School Children to Learn the *NEW* Mathematics

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SINCE 1956 I HAVE BEEN INVOLVED WITH the introduction of various nontraditional mathematical concepts to elementary-school children. These concepts are "nontraditional" in terms of the curriculum ordinarily offered to elementary-school children; they are not necessarily new from a mathematical standpoint. For example, in 1958 and for several years thereafter, Newton Hawley and I experimented extensively on the introduction of geometrical constructions to primary-grade children.[1]

Editor's note. The research reported here has received support from several sources: the National Science Foundation (Course Content Improvement Section), the Cooperative Research Program of the U.S. Office of Education, and the Carnegie Corporation.

[1] For an evaluation by persons outside the project, *see* Thomas Denmark and Robert Kalin, "Suitability of Teaching Geometric Construction in Upper Elementary Grades—A Pilot Study," *The Arithmetic Teacher,* February, 1964, *11,* 73–80.

Beginning in 1959 I turned my attention to the basic arithmetic curriculum, and since that time have been developing the *Sets and Numbers* series as a new basic, but very much enriched, curriculum in elementary-school mathematics.[2] Work on the Sets and Numbers project is continuing, and by this time next year we hope to have completed the preliminary editions for grades 4, 5, and 6.[3] In the present article, however, I shall stress certain special aspects of research with which we have been concerned for the

[2] The initial year of the Sets and Numbers project has been described in Patrick Suppes and Blair A. McKnight, "Sets and Numbers in Grade One, 1959–60," *The Arithmetic Teacher,* October, 1961, *8,* 287–90. The work of subsequent years has been described in Patrick Suppes and Shirley A. Hill, "The Concept of Set," *Grade Teacher,* April, 1962, *79,* 51, 86–90; and Patrick Suppes and Shirley A. Hill, "Set Theory in the Primary Grades," *New York State Mathematics Teacher's Journal,* April, 1963, *13,* 46–53. Volumes for the series from grades K to 3 are now available from L. W. Singer Company, Syracuse, New York.

[3] Some of the test data on the project appear in Suppes and Hill, "Set Theory in the Primary Grades," *op. cit.*

past several years. I have divided this paper into three sections. The first section discusses the work in mathematical logic that we have conducted with gifted fifth and sixth graders since 1960. The second section deals with our current project of vertically accelerating in mathematics a group of 40 gifted first graders for the entire period of their elementary-school education. Finally, the third section describes some of the psychological experiments on mathematical concept formation, which we have been conducting in conjunction with the pedagogical program.

Mathematical Logic in the Elementary School

In modern times, logic has gained depth and breadth as a subject for study. Only in recent years have the systematic relations between logic and mathematics been established and a completely explicit theory of inference formulated which is adequate to deal with all the standard examples of deductive reasoning in mathematics and the empirical sciences.

The concept of axioms and deduction of theorems from axioms is at the heart of all modern mathematics. The purpose of this project has been to introduce the academically gifted elementary-school child to modern mathematics and mathematical methods at a level that is rigorous but simple enough in presentation and context to permit fairly easy comprehension.

A pilot study in the teaching of mathematical logic to a group of selected fifth-grade students was begun at Clifford Elementary School in Redwood City, California, in the autumn of 1960. The class consisted of 25 students selected from the several fifth-grade classes in the school on the basis of their arithmetic achievement scores. The children had been given the arithmetic test during the seventh month of their fourth-grade year; only students who had scored at the sixth-grade level or above at that time were selected for the logic program. The class was taught in 1960–61 and in 1961–62 by members of the project staff.

Eleven classes with a total of 260 fifth-grade students began the study of logic in the fall of 1961; they were taught by their classroom teachers. During the preceding summer, an intensive four-week course was given to ten teachers and three of the school principals. These same teachers were offered a second course in the fall of 1962 for the continuation of their classes during the academic year 1962–63. Eleven classes of sixth graders consisting of 215 students took part in the second year of study.

During this two-year course, the classes met, on the average, three times a week for about 30 minutes. The chapters of the experimental text had the following titles: Chapter 1. "Symbolizing Sentences in Sentential Logic"; Chapter 2. "Logical Inference: Deriving Conclusions from Given Premises Using Standard Rules of Sentential Inference"; Chapter 3. "Truth and Validity: Diagrams of Truth Value, Conditional Proofs, Indirect Proofs, Invalidity, Consistency"; Chapter 4. "Truth Values and Tautologies"; Chapter 5. "Symbolizing Sentences in Predicate Logic: Terms, Predicates, Universal Quantifiers"; Chapter 6. "Universal Specification and Law of Identity"; Chapter 7. "A Simple Mathematical System: Axioms for the Theory of Addition"; Chapter 8. "Universal Generalization, General Theorems."

Two Stanford University logic classes were used as control groups. These classes consisted of students taking elementary mathematical logic at the customary age and educational level. These two university classes met in the autumn quarters of 1961 and 1962.

Two tests were given to the college group of 186 students in the fall of 1961. The first of these tests was also given to all eleven fifth-grade classes (260 students). This test was on the recognition of the logical structure of sentences and beginning work in formal proofs. For the college group, the mean was 91, the median 97. For the fifth-grade group, the mean was 76, the median 83. Seven fifth-grade classes (164 students) took the second test, which dealt primarily with the symbolic manipulations involved in formal proofs within the framework of sentential logic. The two groups had highly comparable scores on this test. For the college group, the mean was 70, the median 72; for the fifth-grade group, the mean was 67, the median 77.

It should be emphasized that the two control groups of Stanford students were themselves a highly select body of undergraduate college students, with an ability range very comparable to that of the selected fifth graders. As the test scores I have cited indicate, the performance of the fifth graders was comparable to that of the college students. More detailed analysis supports this conclusion. From our work on this project, two conclusions of some significance emerge:

1. The upper quartile of elementary-school students can achieve a significant conceptual and technical mastery of elementary mathematical logic. The level of mastery is 85 to 90 per cent of that achieved by comparable university students.

2. The more dedicated and able elementary-school

teachers can be given sufficient training in five or six semester hours of instruction to teach classes in elementary mathematical logic. It is important that a teacher-training program of no more than six semester hours be very closely geared to the actual program of instruction that the teacher will follow in his own classroom.

Finally, in the summer of 1963 we selected the ablest students from the initial group of 260 and gave them an additional intensive course for four weeks. In the last part of this program, the students were given Boolean algebra in axiomatic form and elementary topology. With their prior training in formal logic, they found it extremely easy to master the technique of thinking intuitively about these new mathematical systems and writing down appropriate proofs of elementary theorems. Our work during this summer suggests that any able group of students who received a broad mathematical training in the last years of elementary school is prepared for an intensive and rich mathematical curriculum during the six years of secondary school.[4]

Vertical Acceleration of Gifted First Graders

Beginning with the current academic year, we have undertaken an intensive accelerated program of mathematics for a group of gifted first graders selected from four elementary schools in the Palo Alto Unified School District. As the first step in selection, the New York Test of Arithmetical Meaning was given to all entering first-grade students in the four schools. Students ranking above the 70th percentile on this test of first-grade achievement were then tested more intensively; most of them were given individual Stanford-Binet's. This procedure resulted in the selection of 40 children having IQ's ranging from 122 to 166 with a mean of 137. At the present time, these children are being taught mathematics in the four schools in small groups of 9 to 12 pupils. The classes meet four days a week with each session lasting not more than 35 minutes.

For each of the 40 students, we are collecting data on the number of problems worked and the errors made each day. During the first seven weeks, a typical child in the *bottom* quartile of the group worked approximately 2,500 problems. In terms of the *Sets and Numbers* mathematics series, the topics covered in working these problems are: sets; notation for sets; identity of sets; union of sets; number of things in a set; addition of numbers; place-value system of notation for numbers; difference of sets; subtraction of numbers; applications: telling time, linear and liquid measures, changing money; recognition of triangles, quadrilaterals, pentagons, and circles; inside and outside of a simple closed figure; figures of the same shape and size; figures of the same shape but of different sizes; equilateral triangles; perimeter of simple polygons. During the same period, the children in the top quartile of the group also covered the introduction of the numbers 11 to 19, with considerable practice on their properties, the place-value notation to hundreds, the concept of *subset,* and the corresponding numerical concept of *less than*—a total of more than 500 additional problems.

At the present rate of progress, most of the students in the selected group will complete our already enriched second-grade curriculum by the end of the first grade. At the same time, they will also have worked through Book 1 of *Geometry for Primary Grades* as supplementary material. Children in the top quartile will have covered up to half of the third-grade curriculum in the Sets and Numbers program. We are, of course, impressed with the rapid progress that these children are making. The students in the top quartile have been working at an enormous rate; during the second week, for example, they averaged more than 150 problems a day, and during the first seven weeks they did not fall below the rate of 50 problems a day during any single week. Naturally, we expect this rate to slow down as they advance in the curriculum and come to more difficult material. It is, however, indicative of the speed with which brighter children can cover modern mathematical topics designed for a standard curriculum and absorption by all children.

Figure 1 shows the rate of progress during the first seven weeks of the top four students in the group of 40. The decline in rate from the second week to the fifth week is a result of the increasing difficulty of the material in *Sets and Numbers,* Book 1B. The increase in rate from the fifth to the seventh week is due to the fact that at this point the children had begun *Sets and Numbers,* Book 2A, which is the first half of the second-grade course; the early part of this book spends a good deal of time reviewing the concepts introduced in the first grade. The error rate for the four students represented in Figure 1 averaged less than 3 per cent for the seven weeks.

[4] Further details on this mathematical-logic project are contained in the final report, "Experimental Teaching of Mathematical Logic in the Elementary School," written by the author and Frederick Binford, on deposit with the Cooperative Research Program, U.S. Office of Education

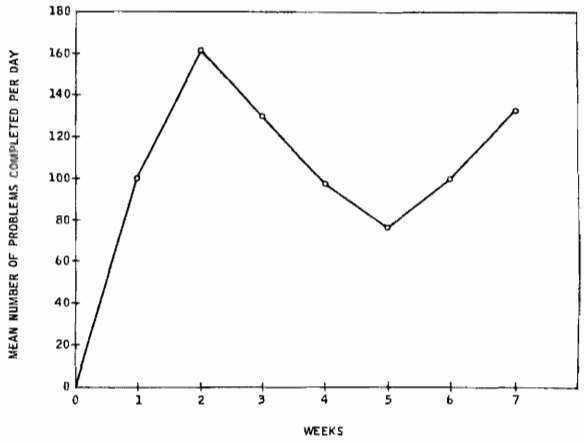

FIGURE 1

Before completing this part of the discussion, I want to make the point that the ability to learn the new mathematics concepts is not restricted to gifted children. I have attempted to show how rapidly gifted children advance through the new concepts organized in terms of the standard curriculum aimed at the average child. We also have in our project files data on achievement records with a class of mentally retarded students ranging in age from eight to fourteen years who worked with the first-grade books. Their achievement scores indicated that they were mastering the basic concepts and were showing an ability to move ahead, though at a slow pace. It is perhaps already obvious that with proper pacing and presentation all students are able to learn the concepts of modern mathematics. Probably the more important point to be drawn from our own research is that the brighter children—those, let us say, in the upper quartile—are able to cover a great many more mathematical concepts and to learn a good deal more mathematics during their elementary-school years than we have been willing to admit in the recent past.

Research on Mathematical Concept Formation

Since 1959 we have also been conducting in our laboratories at Stanford an extensive program of psychological research on mathematical concept formation in children. The evidence of these experiments, which I shall not attempt to report in detail here, clearly indicates that many of the concepts of mathematics not traditionally taught in the elementary school are easily within the range of the average child. A few examples will illustrate this point.

In an experiment conducted by Erline Stoll, one group of kindergarten children was taught to recognize triangles, quadrilaterals, and pentagons; and another group was taught to discriminate between acute, right, and obtuse angles. Most of the subjects were able to satisfy the rather strong criterion of nine successive correct responses, even though only minimal instructional aids were provided by the experimenter.[5]

In another experiment, five- and six-year-old subjects were taught to recognize the notational pattern of the numbers 4 and 5 in the binary number system. Most of the children were able to master this relatively simple concept in one experimental session.[6]

We have also experimented extensively with the learning of the concept of identity of sets and equivalence of sets. First-grade children who had had no systematic instructions in set concepts made very good progress in a tightly controlled and tightly structured experimental setup in which only minimal instructions were given by the experimenter.[7]

Shirley Hill made an extensive empirical study of the logical abilities of children. She gave a test instrument consisting of 100 items to 270 children ranging in age from six to eight years (first, second, and third grades). Each of the 100 items, presented orally, consisted of two or three verbal premises plus a conclusion formulated as a question. The subject was asked to affirm or deny the conclusion as presented. The 100 items were divided equally between positive and negative answers. The first part of the test consisted of 60 items drawn from sentential logic. Each conclusion or its negation followed from the given premises by the sentential theory of inference. The second part consisted of 40 items drawn from predicate logic, including 13 classical syllogisms. The predicate-logic items also included inferences using two-place predicates together with existential quantifiers.

Every attempt was made to construct the items in such a way that the omission of one premise would make it impossible to draw the correct conclusion.

[5] *See* Erline A. Stoll, "Geometrical Concept Formation in Kindergarten Children." Doctoral dissertation, Stanford University, 1962.

[6] *See* Patrick Suppes and Rose Ginsberg, "Application of a Stimulus Sampling Model to Children's Concept Formation with and without Overt Correction Responses," *Journal of Experimental Psychology*, April, 1962, *63*, 330–36.

[7] *See* Patrick Suppes and Rose Ginsberg, "A Fundamental Property of All-or-None Models, Binomial Distribution of Responses Prior to Conditioning, with Application to Concept Formation in Children," *Psychological Review*, March, 1963, *70*, 139–61.

To provide a behavioral check on this aspect of the items, a base-line group of 50 subjects was given the test with the first premise of each item omitted. This group got an average of 52.02 per cent of the items correct, which does not significantly differ from chance.

The major results of the complete test follow. The first graders got 71.18 per cent of the items correct; the second graders got 79.54 per cent of the items correct; and the third graders got 85.58 per cent correct. This study provides substantial evidence that even six-year-old children are able to deal very effectively with verbal premises that call for hypothetical reasoning and are by no means limited to "concrete" operations.[8]

The program of psychological experimentation as an extension and adjunct to our pedagogical program is continuing. It is fair to say that we do not yet have a clear idea of the full potentialities of below-average, average, and bright children for learning the concepts of mathematics. The problem is a complex one. I am sure that we shall not be able to solve it definitively in the next few years. We do hope, however, to continue to make contributions to its solution.

[8] *See* Shirley A. Hill, "A Study of the Logical Abilities of Children." Doctoral dissertation, Stanford University, 1961.