The Stanford Institute for Mathematical Studies in the Social Sciences

APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY

Reprint No. 34

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Reprinted from
Journal of Experimental Psychology
Vol. 61, No. 2, 1961

Reprinted from JOURNAL OF EXPERIMENTAL PSYCHOLOGY,
Vol. 61, No. 2, February, 1961
Printed in U. S. A.

TEST OF STIMULUS SAMPLING THEORY FOR A CONTINUUM OF RESPONSES WITH UNIMODAL NONCONTINGENT DETERMINATE REINFORCEMENT [1]

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In the last decade a large number of quantitative experimental studies of simple learning have been made for cases in which the reinforcement schedule is probabilistic and the number of possible responses is finite. Both stimulus sampling and linear models have been used extensively to analyze these studies. There have been, as far as we know, no previous attempts to extend these models to situations which permit a continuum of possible responses. Such is the aim of the present study, which may be briefly described as follows.

The S is told that his task on each trial is to predict by means of a pointer where a spot of light will appear on the circumference of a circle; S's responses are his pointer predictions. At the end of each trial the "correct" position of the spot is shown to S; this is the reinforcing event for the trial. The response x and the reinforcement y vary continuously along a circle from 0 to 2π. The most important variable controlled by E is choice of a particular probability distribution of reinforcement. In the present study reinforcement is noncontingent, i.e., the probability distribution of reinforcements is independent of S's responses and the same distribution is used on every trial. Moreover, the reinforcement is determinate because exactly one reinforcement y occurs on each trial. The purpose of a model is to predict the probability distribution of responses.

To analyze this situation we shall apply the stimulus sampling and linear models developed in Suppes (1959, 1960). Because the stimulus sampling model provides a more deatiled account of the psychological process of learning than does the linear model, and because its mathematical development is somewhat simpler, we shall concentrate mainly on it. However, the theoretical results considered here also hold in the linear model.

Stimulus sampling model.—The basic theory derives from Estes (1950). Roughly speaking, this theory runs as follows. On each trial S samples a set of stimuli each of which is conditioned to exactly one response. The probability that S will make a given response is the proportion of sampled stimuli conditioned to that response, and all the sampled stimuli become conditioned to the response which is reinforced. Thus, theoretically the learning process is described in terms of the states of the conditioning of the stimuli, these states not being directly observable. For many experiments it has proved desirable to assume that exactly one stimulus is sampled on each trial and that the probability is not necessarily one that the sampled stimulus becomes conditioned to the reinforced response. We shall designate as θ the probability that reinforcement is *effective* and shall postulate the usual independence of path assumptions for this probability. In our context θ plays

[1] This research was supported by the Rockefeller Foundation and the Group Psychology Branch of the Office of Naval Research.

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the role of the learning parameter. To simplify considerably the mathematics we assume there is exactly one stimulus element that is available for sampling on every trial and is therefore sampled with probability 1.0. Most of the theoretical results given here actually do not depend on this one-element assumption.

For a continuum of responses, one fundamental modification in the theory just described seems called for. The assumption that each stimulus is conditioned to exactly one response seems unsound, and we replace it with the assumption (Suppes, 1960) that the conditioning of each stimulus is *smeared* over a certain interval of responses, possibly the whole continuum. We represent the conditioning of any stimulus by a *smearing distribution*, $K(x; y)$. It is postulated that parameter y, the mean of the distribution, takes on the value of the point of reinforcement when the stimulus is sampled and reinforcement is effective. In mathematical language, the basic learning process is a continuous-state discrete-trial Markov process in parameter y. On each trial, with probability θ, y moves to the new point of reinforcement, and with probability $1 - \theta$ it stays in the same place.

Various intuitive arguments can be given to support the representation of conditioning by a smearing distribution. It might seem that the best one would be based on well-known psychophysical facts about the impossibility of making exact discriminations among a continuum of choices. From this standpoint the smearing distribution would arise from the perceptual inability of the organism to distinguish exactly a given point on the continuum. However, the empirical estimates of the smearing distribution, which are discussed in detail later, yield a variance for the distribution which is far in excess of that which can be justified on purely psychophysical grounds. It seems likely that a better argument can be put together in terms of ideas of response generalization, but to make these ideas precise would require theoretical developments beyond the scope of the present paper.

We now summarize the results which are proved in Suppes (1960) and which we shall use in the analysis of data. In the form given, the equations hold only for a noncontingent reinforcement distribution, $F(y)$.[3] Let $[a, b]$ be the interval of possible responses; let $r_n(x)$ be the mean response density averaged over Ss on trial n; and let $k(x; y)$ be the smearing density of the single stimulus. Then we have the recursion

$$r_{n+1}(x) = (1 - \theta)r_n(x) + \theta \int_a^b k(x; y)f(y)dy \quad [1]$$

and thus for $r(x) = \lim_{n\to\infty} r_n(x)$, the important asymptotic relation

$$r(x) = \int_a^b k(x; y)f(y)dy \quad [2]$$

Multiplying both sides of Equation 1 by x^r, using Equation 2 and integrating with respect to x, we obtain as the recursion for the rth raw moment $\mu_{r,n}$ of $r_n(x)$

$$\mu_{r,n+1} = (1 - \theta)\mu_{r,n} + \theta\mu_r \quad [3]$$

and this difference equation may be solved explicitly to yield

$$\mu_{r,n} = \mu_r - [\mu_r - \mu_{r,1}](1 - \theta)^{n-1} \quad [4]$$

where $\mu_r = \lim_{n\to\infty} \mu_{r,n}$. Let $\mathcal{E}_n$ and σ_n^2 be the mean and variance of $r_n(x)$; and $\mathcal{E}$ and σ^2 the mean and variance of the asymptotic distribution $r(x)$. Then the recursion for σ_n^2 is as follows:

$$\sigma_{n+1}^2 = (1 - \theta)\sigma_n^2 + \theta\sigma^2 + \theta(1 - \theta)(\mathcal{E}_n - \mathcal{E})^2 \quad [5]$$

We use these last two equations to estimate θ. It is important to note that if the initial and asymptotic mean response densities are estimated directly from the data, then Equations 3–5 may be used independently of any particular choice of a smearing distribution $K(x; y)$.

Also independent of the choice of a smearing distribution is the prediction from Equation 2 that for a unimodal re-

[3] We use capital letters for distributions and lower case letters for the corresponding densities.

inforcement distribution the dispersion of the response distribution is greater than that of the reinforcement distribution. This particular prediction is an excellent gross test of the theory, for the rational strategy in terms of statistical decision theory is to make all responses at the mean of the reinforcement distribution and thus to have an asymptotic response distribution with zero variance.

It may also be shown that the variance of the asymptotic mean response distribution is approximately the variance of any long sequence of an individnal *S*'s responses. This means that the prediction about the dispersion of responses is a prediction about individual behavior, not simply that of the group of *S*s.

In like fashion it follows from Equation 2 that if the reinforcement and smearing distributions are symmetric, then the response distribution is also symmetric. Again this is a prediction that should hold for individual *S*s as well as for group averages.

The method used for estimating the smearing distribution is discussed below.

METHOD

Subjects.—Twenty-three male and thirty-seven female Stanford undergraduates, obtained from the student employment bureau and through personal contact, were employed as *S*s. Each *S* was paid $1.25 for the 1-hr. experimental session.

Apparatus.—From *S*'s side the apparatus appeared as a vertical blue plastic screen 6-ft. square mounted 8 in. above the floor. At either side hung heavy white curtains which blocked *S*'s view of the area beyond the screen. A small amount of indirect illumination came from behind the curtains.

The translucent plastic was stretched across a 4-in.-deep wooden box. A $\frac{3}{8}$-in. steel shaft supported horizontally in a bearing mounted on the back of the box projected 21 in. on *S*'s side of the screen. Within the enclosure, light sources supported on a counterbalanced arm affixed to the shaft were arranged to project a $\frac{1}{8} \times \frac{3}{4}$-in. bar of red light onto the screen at either 1.25 ft. or 2.5 ft. from the shaft. Light sources were 6.3-v. pilot lights (RCA No. 47) covered by red plastic domes. The short dimension of the bars of light was perpendicular to the supporting arms.

By turning a knob on the end of the shaft *S* could make his prediction by positioning the light bar at any point on the 2.5-ft.-diameter circle about the axis of the shaft (or 5.0-ft.-diameter circle, depending on *S*'s group). The *S* was seated at a classroom desk chair so that the shaft was about at eye level. An optional arm rest was mounted on the chair arm and *S* was instructed to use either hand and to change hands or position whenever he wished.

A 23-in.-diameter scale with 400 equally spaced divisions was mounted behind the enclosure to indicate the location of the light. The entire scale could be rotated to change the physical location of the zero point. By turning a knob on his end of the shaft, *E* could position the light bar to indicate reinforcement location.

A 40-w. fluorescent light tube mounted over the scale furnished the only illumination in the experimental room.

Procedure.—After acquainting *S* with the method of operating the equipment, *E* read the following instructions.

> In various military situations the most effective operators of radar fire-control and target-locating devices are those who most frequently anticipate the approximate location of the next target. We are studying how this skill is developed.
>
> If you are holding the knob, please release it and I will show the target. Do you see the bar of red light that is moving around the screen? The target may be anywhere on the circle that the light is moving in. (Stop light at top of circle.) You will use the knob in front of you to move the light to where you think the target will be. Try it. O.K.? Now, when you have the light where you want it, release the knob and say, "Mark!" Then I will move the light to the target position and say, "Target!" As soon as you see where the target is take the knob again and move the light to where you think the next target will be, release the knob, and say, "Mark!" Then I will show you the target position as before. Because time is limited, you will only have 5 sec. to make your decision each time. You will seldom need this much time, but when you exceed the limit I will say, "Time!" and you will then have 1 sec. to position the light.
>
> At first, you will have to guess about the target location, but, with practice, your predictions should improve. The factors which determine target location are very complex, and you will find that target position changes quite erratically. You will seldom

be able to make an exact prediction but errors should reduce so that much of the time your predictions are close to the vicinity of the target.

As soon as questions had been answered by paraphrasing the instructions 300 trials were run without interruption. The average rate was 6 trials per minute.

Design.—There were two groups of 20 *S*s, one group using the 2.5-ft. circle and the other the 5.0-ft. circle.

Thirty 300-trial reinforcement sequences were computed using the triangular density shown in Fig. 2 below. By random choice 30 equally spaced divisions of the circle, starting from arbitrary fixed physical zero, were assigned without repetition as scale zero points for the separate reinforcement sequences. Each reinforcement sequence was employed for one *S* using the 2.5-ft. circle and one *S* using the 5.0-ft. circle. The *S*s were assigned to circle sizes randomly with the restriction that matched reinforcement sequences were run successively.

Results and Discussion

Response histogram variances over trials.—Table 1 shows the means and variances of the response histograms averaged over *S*s in each group in blocks of 50 trials. The closeness of the means in the first block of trials to the mean, π, of the reinforcement distribution is due to the randomly selected scale origin for each *S*. The randomization of the physical position of the scale origin guarantees a uniform distribution on Trial 1 as the number of *S*s becomes large, and the mean of the uniform distribution on the interval 0 to 2π is π. Consequently we have used the change in variance of the response histogram to estimate learning parameter θ.

The rates of decrease in the variances indicate both groups were close to their asymptotic variances at the end of the first 100 trials, if we take the variances of the last two blocks of trials as an estimate of the asymptotic variances. In Fig. 1 are plotted the variances of the response histograms averaged over *S*s in each group in blocks of 10 trials for the initial 100 trials. Because of its small magnitude, the term $\theta(1-\theta)(\mathcal{E}_n - \mathcal{E})^2$ in Equation 5 was omitted, yielding the recursion

$$\sigma_{n+1}^2 = (1-\theta)\sigma_n^2 + \theta\sigma^2$$

which has the solution

$$\sigma_n^2 = \sigma^2 - (\sigma^2 - \sigma_1^2)(1-\theta)^{n-1}$$

Averaging over blocks of 10 trials, we have, for $i = 1, 2, \cdots, 10$

$$\sigma_i^2(\theta) = \tfrac{1}{10}\sum_{v=10i-9}^{10i}\sigma_v^2 = \sigma^2 - \frac{1-(1-\theta)^{10}}{10\theta} \times (\sigma^2-\sigma_1^2)(1-\theta)^{10(i-1)} \quad [6]$$

TABLE 1

Observed Means and Variances of Response Density for Blocks of 50 Trials

Block	2.5-Ft. Circle		5.0-Ft. Circle	
	Mean	Var.	Mean	Var.
1	.990	.258	1.037	.260
2	.991	.223	.998	.229
3	.991	.228	.998	.219
4	.996	.224	.990	.210
5	1.011	.225	.984	.215
6	.990	.219	1.001	.197

Note. All table entries are observed values divided by π for means and π^2 for variances.

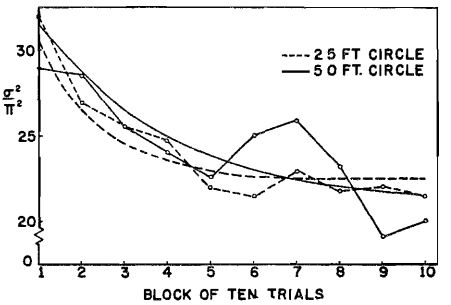

Fig. 1. Observed variances of $r_n(x)$ in blocks of 10 trials for initial 100 trials and variance curves fitted by least squares.

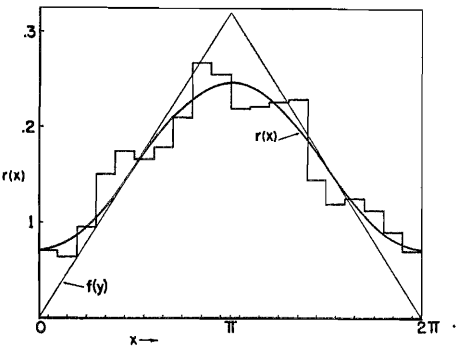

FIG. 2. Response histogram for 2.5-ft. circle and fitted response density for last 200 trials.

Equation 6 and the observed variances σ_i^2(obs) were used to find by the method of least squares the value of θ which yields a curve which best fits, for each circle, the observed variances plotted in Fig. 1. That is,

$$L(\theta) = \sum_{i=1}^{10} [\sigma_i^2(\text{obs}) - \sigma_i^2(\theta)]^2$$

was minimized. Because the derivative of $L(\theta)$ with respect to θ is a complicated function, the minimum was not found by analytical methods but $L(\theta)$ was computed for each circle and the minimum θ approximated to four decimal places. For the 2.5-ft. circle, the estimated θ is .0650, and for the 5.0-ft. circle the estimated θ is .0325. The fitted curves based on the estimated θ's are shown in Fig. 1. These computations were based on the assumption that $\sigma_1^2 = \frac{\pi^2}{3}$, the variance of the uniform distribution on the interval $[0, 2\pi]$. The fit to the data from the 2.5-ft. circle is the better of the two. This is because of the unexpected increase in variance following Block 5 for the 5.0-ft. circle. The increase is contrary to the theory and seems to be too large to be due to sampling error. The estimates of θ are surprisingly close to those obtained by Estes and Straughan (1954) from the mean learning curve for two-choice noncontingent data.

Asymptotic response histograms.— Figures 2 and 3 present for the two groups of *S*s response histograms in class intervals of $.1\pi$ for the last 200 trials. The triangular reinforcement density $f(y)$ is also shown in both figures.

The reinforcement density $f(y)$ has a mean of π and a variance of $.167\pi^2$. The response histogram for the group of *S*s using the 2.5-ft. circle has a mean of $.997\pi$ and a variance of $.224\pi^2$. The response histogram for the group using the 5.0-ft. circle has a mean of $.993\pi$ and a variance of $.210\pi^2$.

Individual data are presented in Table 2. The mean, variance, and third central moment are given for each individual histogram for the last 200 trials. These moments for the 200 reinforcements are also tabled. Note that for each pair of *S*s, one for the 2.5-ft. circle and one for the 5.0-ft. circle, the moments of only one reinforcement histogram are tabulated because the two members of the pair received exactly the same sequence of reinforcements.

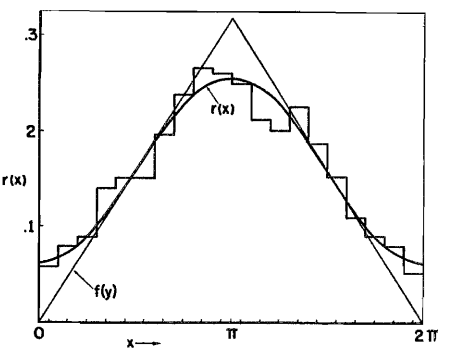

FIG. 3. Response histogram for 5.0-ft. circle and fitted response density for last 200 trials.

TABLE 2

INDIVIDUAL OBSERVED MEANS, VARIANCES, AND THIRD MOMENTS FOR LAST 200 TRIALS

S	Reinforcement Distribution			2.5-Ft. Circle			5.0-Ft. Circle		
	Mean	Var.	Third Moment	Mean	Var.	Third Moment	Mean	Var.	Third Moment
1	.95	.163	.000	1.01	.277	.006	1.00	.209	.003
2	1.03	.159	.002	1.03	.180	.000	.99	.253	.004
3	.98	.155	.002	1.08	.185	.007	1.02	.131	.005
4	.97	.173	.010	.97	.304	.016	1.03	.240	.004
5	.98	.164	.004	.96	.167	.000	1.07	.178	.011
6	1.04	.147	.000	1.01	.184	.013	.95	.128	.003
7	.98	.176	.005	1.10	.220	.001	1.10	.152	.015
8	1.03	.179	.002	.99	.149	.005	1.01	.197	.012
9	.97	.163	.000	.93	.240	.016	.96	.224	.001
10	.94	.170	.007	.97	.268	.031	.92	.221	.034
11	1.00	.156	.006	1.03	.305	.013	1.05	.210	.001
12	.98	.157	.001	1.04	.250	.019	1.01	.222	.005
13	1.00	.174	.008	.89	.138	.008	.97	.337	.021
14	.99	.175	.004	1.02	.202	.003	.96	.169	.002
15	1.01	.157	.003	.94	.160	.011	.91	.301	.006
16	1.01	.179	.002	.97	.209	.032	.92	.248	.020
17	.97	.181	.010	.96	.297	.003	.94	.256	.022
18	1.03	.214	.004	1.03	.228	.000	1.13	.262	.050
19	.95	.185	.009	.95	.225	.031	.86	.219	.040
20	.99	.152	.004	.94	.223	.009	.99	.146	.004
21	.97	.150	.006	1.02	.240	.006	.99	.188	.000
22	1.00	.166	.006	1.01	.172	.011	.96	.130	.006
23	1.01	.149	.007	.95	.204	.007	.97	.230	.007
24	1.01	.179	.002	.97	.203	.016	1.00	.291	.017
25	1.08	.149	.001	1.05	.168	.009	1.02	.089	.005
26	1.01	.171	.016	1.01	.291	.005	.99	.230	.001
27	1.02	.148	.008	1.08	.231	.017	.99	.170	.010
28	1.00	.169	.000	.95	.381	.018	1.04	.270	.003
29	1.00	.127	.006	1.00	.256	.002	1.01	.177	.003
30	1.02	.159	.003	1.09	.083	.004	1.04	.136	.012

Note. All table entries are observed values divided by π for means, π^2 for variances, and π^3 for third central moments.

Putting aside for the moment detailed quantitative questions, these individual data qualitatively support the fundamental theory in three important respects.

First, the means of the individual histograms are all quite close to the mean of the reinforcement distribution, as is predicted by the theory. In the case of the 2.5-ft. circle, only 3 *S*s deviate from the theoretical mean of 1.0π by more than does the most deviant reinforcement histogram (that for *S* 25). In the case of the 5.0-ft. circle there are only 5 such *S*s. The deviation from the theoretical mean of no *S* in either group is more than twice that of this most deviant reinforcement histogram (whose deviation is 15°). These results would seem to show that the reinforcement conditions obtaining in the experiment easily dominated any a priori positioning habits.

Secondly, the variances of the individual histograms are generally larger than those of the corresponding reinforcement histograms. In particular, this is true for 27 of the 30 *S*s on the 2.5-ft. circle and 22 of the 30 *S*s on the 5.0-ft. circle. This result shows rather clearly that there was no general tendency for individual *S*s to adopt the rational strategy of statistical decision theory,

i.e., asymptotically to respond very close to the mean of the reinforcement distribution.

Thirdly, the relatively small magnitudes of the individual third central moments indicate a high degree of symmetry in the individual histograms. The third moments for 21 *S*s on the 2.5-ft. circle and 23 on the 5.0-ft. circle are larger than those of the corresponding reinforcement histograms, but none seems excessive; only 3 of the 60 *S*s have a third moment more than twice the largest third moment of the 30 reinforcement histograms.

It seems to us that these results for individual *S*s are sufficiently encouraging to warrant quantitative investigations of the extent to which individual *S*s are satisfying the model. Such investigations will require many more trials than were obtained in the present study. For this reason the remainder of the analysis in this paper is restricted to group data.

Estimation of smearing distributions. —The noncontingent triangular reinforcement distribution shown in Fig. 2 and 3 is analytically defined by the density

$$f(y)=\begin{cases} \dfrac{y}{\pi^2} & 0<y\leq\pi \\ \dfrac{1}{\pi^2}(2\pi-y) & \pi<y\leq 2\pi \end{cases} \qquad [7]$$

For the circular display it is reasonable to assume that the smearing distribution about a point of reinforcement, y, is symmetric, and therefore $k(x;y) = k(t)$, where $t = x - y$. For detailed analysis of data we restrict ourselves to two general forms for the smearing distribution—*uniform* distributions and *symmetric beta* distributions, with the free parameter of the distribution to be estimated from the data by use of Equation 2. Two things have motivated this choice, namely, the mathematical simplicity of both distributions and the fact that as m increases the symmetric beta distribution closely resembles the normal distribution, and thus has the bell shape which seems intuitively appropriate for a smearing phenomenon.

The *uniform smearing* density with parameter a is defined by

$$k(t) = \begin{cases} \dfrac{1}{2a} & |t|\leq a \\ 0 & a<|t|\leq\pi \end{cases} \qquad [8]$$

where $-\pi \leq t \leq \pi$ and $0 < a < \frac{\pi}{2}$. Combining Equations 2, 7, and 8, and integrating, we have for the asymptotic response density:

$$r(x) = \begin{cases} \dfrac{1}{2a\pi^2}(a^2+x^2) & 0<x\leq a \\ \dfrac{x}{\pi^2} & a<x\leq\pi-a \\ \dfrac{1}{2a\pi^2}(2a\pi-a^2-(x-\pi)^2) & \pi-a<x\leq\pi \end{cases} \qquad [9]$$

for $0 < x \leq \pi$, and because $r(x)$ is symmetric about π, it is sufficient to know its values for $0 < x \leq \pi$. For estimation of a, it is desirable to compute the variance of $r(x)$ as defined by Equation 9, which is:

$$\sigma_{U(a)}{}^2(r(x)) = \frac{\pi^2}{6}\left[1+\left(\frac{a}{\pi}\right)^2\left(2-\frac{a}{\pi}\right)\right] \qquad [10]$$

where the subscript $U(a)$ denotes the uniform smearing distribution with parameter a.

The *symmetric beta smearing* density with parameter m is defined by:

$$k(t) = \frac{1}{\pi B(m, \frac{1}{2})}\left(1 - \frac{t^2}{\pi^2}\right)^{m-1} \qquad [11]$$

for $-\pi < t < \pi$ and $B(m, \frac{1}{2})$ the usual beta coefficient. (Because of the form of the reinforcement density $f(y)$, Equation 2 must be written as

$$r(x) = \frac{1}{\pi^2}\left\{\int_{-\pi}^{x-\pi} k(t)(2\pi - x + t)dt + \int_{x-\pi}^{x} k(t)(x - t)dt + \int_{x}^{\pi} k(t)(t - x)dt\right\} \qquad [12]$$

and thus in considering the beta smearing density it is necessary to restrict the exponent $m - 1$ to integer values in order to integrate Equation 11, which restriction turns out not to be serious.) Combining now Equations 2, 7, and 11, and again integrating, we obtain for $0 < x < \pi$:

$$r(x) = \frac{1}{\pi}\left\{\frac{(\pi^2 - x^2)^m - (\pi^2 - (x - \pi)^2)^m}{m\pi^{2m}B(m, \frac{1}{2})} + 1 - \frac{x}{\pi} + \frac{2}{B(m, \frac{1}{2})}\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}(-1)^\nu \frac{x^{2\nu+2} - (x - \pi)^{2\nu+2}}{(2\nu + 1)\pi^{2\nu+1}}\right\} \qquad [13]$$

For the variance of $r(x)$ as defined by Equation 13, we have:

$$\sigma_{\beta(m)}{}^2(r(x)) = \pi^2\left\{\frac{2}{2m + 1} + \frac{3}{2(2m + 3)(2m + 1)} - \frac{2}{B(m, \frac{1}{2})}\left[\frac{1}{m(m + 1)} - 2\sum_{\nu=0}^{m-1}\binom{m-1}{\nu}(-1)^\nu \frac{B(2\nu + 3, 3)}{2\nu + 1}\right]\right\} \qquad [14]$$

Asymptotic response densities, as given by Equation 9 for the uniform smearing density and Equation 13 for the beta smearing density, were fitted by the method of moments to the response histograms shown in Fig. 2 and 3. In the present situation this amounted to estimating the values of parameters a and m, respectively, which determine asymptotic response densities whose variances (Equations 10 and 14) are equal to the variances of the histograms. The variances of the response histograms were computed from the individual observations and not from the frequency functions for the class intervals. The histograms and the fitted response densities both show the smoothing and spreading effects which are to be expected if the notion is correct of a distribution which smears the conditioning of the sampled stimulus.

As already stated, the response histogram for the group of *S*s using the 2.5-ft. circle has a mean of $.997\pi$ and a variance of $.224\pi^2$. In order to fit this response histogram, if we take a uniform smearing density with $a = .475\pi$, we obtain an asymptotic response density with mean of π and variance of $.224\pi^2$. And if we take $m = 6$, we obtain for the beta smearing density, an asymptotic response density with mean of π and variance of $.222\pi^2$. The two response densities resulting from the two smearing densities are so nearly identical, it was possible to plot only one response density in Fig. 2; the one selected is that arising from

the beta smearing density. For numerical comparison, the maximum difference between the two response densities occurs at $x = 0$, for which $r_U(x) = .076$ and $r_\beta(x) = .071$. Goodness of fit tests for the response densities are given below.

The response histogram for the 5.0-ft. circle has a mean of $.993\pi$ and a variance of $.210\pi^2$. In order to fit this histogram, if we take a uniform smearing density with $a = .405\pi$, we obtain an asymptotic response density with mean of π and variance of $.210\pi^2$. And if we take a beta smearing distribution with $m = 8$, we get an asymptotic smearing density with mean of π and variance of $.211\pi^2$. Again the two response densities are so similar it was possible to plot only $r_\beta(x)$ in Fig. 3. At $x=0$, $r_U(x)=.064$ and $r_\beta(x)=.063$.

Because of its greater circumference the larger circle permits finer discrimination; thus, when measurements are made in radians, the variance of the response density and that of the corresponding smearing density are less for the larger of the two circles. The four fitted smearing densities are plotted in Fig. 4.

Figure 5 presents the theoretical variance of the asymptotic response density as a function of parameter a of the uniform smearing density and also as a function of parameter m of the beta smearing distribution. The empirical variances of the response histograms for the two circles appear as horizontal lines in the figure. The figure shows that the use of integer values for parameter m of the beta smearing density is sufficiently sensitive for good estimation of the asymptotic response density. It should be noted that when the variance of the smearing distribution goes to zero, the asymptotic response density is identical with the noncontingent reinforcement density.

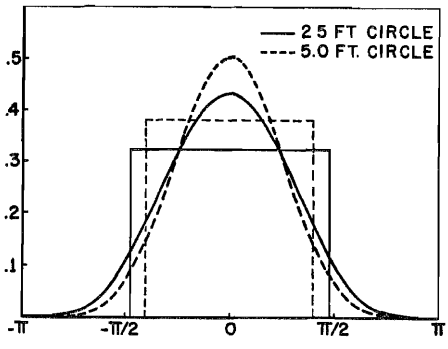

FIG. 4. Fitted smearing densities.

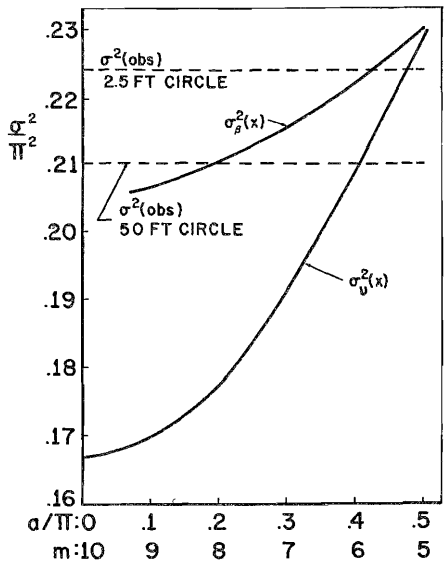

FIG. 5. Theoretical variance of $r(x)$ for uniform smearing density with parameter a and beta smearing density with parameter m.

Prior to detailed calculations it was not anticipated that the exact form of the smearing distribution would affect so slightly the form of the asymptotic response distribution. This is a highly encouraging result for the investigation of more complicated reinforcement distributions which pose increasingly difficult mathematical problems, for in most cases the possibility of assuming a uniform smearing distribution leads to considerable formal simplifications.

On the other hand, the near identity of the response distributions arising from the two different smearing distributions implies that there is little possibility of identifying from response histograms the general form of the smearing distribution. Theoretically Equation 2 may be regarded as an integral equation, and when $k(x; y)$ is taken to be symmetric, Equation 2 may be easily transformed into a standard integral equation of Fredholm type with the reinforcement density $f(y)$ as the kernel. Given $f(y)$ and an analytical function $r(x)$ it is then possible to determine the class of functions $k(x; y)$ which satisfy the integral equation. However, as the discussion of Fig. 1 and 2 has indicated, response densities of quite different analytical form fit the empirical response histograms equally well and the choice between them is empirically arbitrary. It remains to be seen whether other properties of the model, which have not been considered here, will yield a practical method for identifying the form of the smearing distribution.

Goodness of fit of response densities. —Inspection of Fig. 2 and 3 suggests that the response densities fit the response histograms better than the triangular reinforcement distribution does, and that the over-all fit is reasonably good. Because of the importance of this goodness of fit for evaluating the validity of the theory, it is pertinent to examine it more closely. The obvious procedure is to apply a χ^2 test, but the difficulty is that the 6,000 observations on which each histogram and fitted response density are based are not statistically independent, as is required for a χ^2 test. However, it is known that for stationary stochastic processes observations sufficiently spaced in trials are statistically independent. Following this line of thought, we considered responses on every fifth trial for each S starting with Trial 102. To test the statistical independence of these responses, we divided the circumference of the circle into six class intervals with approximately equal relative frequencies for the 2.5-ft. circle data. In terms of these six class intervals we then tested the null hypothesis that the 1200 responses were drawn from a zero-order Markov process (i.e., are statistically independent) against the alternative hypothesis that they were drawn from a first order Markov process. The appropriate χ^2 test is given in Anderson and Goodman (1957) and has been used repeatedly on learning data by Suppes and Atkinson (1960). For the 2.5-ft. circle, $\chi^2 = 32.97$, which with $(6 - 1)^2 = 25$ df has a significance level of $P = .20$. For the 5.0-ft. circle $\chi^2 = 30.66$, which with the same number of df has $P = .24$. For purpose of comparison the same test was run on the reinforcements occurring on the immediately preceding trial, i.e., Trials 101, 106, etc. Here $\chi^2 = 24.95$ which has $P = .49$. With 1,200 observations the power of this χ^2 test is high, so that with the obtained significance levels it is reasonable to accept the null hypothesis that the responses are statistically independent.

Using then the 1,200 statistically independent responses on each circle, the following results were obtained. First, the null hypothesis that the response data fit the reinforcement distribution was tested. On the basis of dividing the circle into 30 class intervals of approximately equal frequency for the 2.5-ft. circle data, the rejection of the null hypothesis was decisive. For the 2.5-ft. circle $\chi^2 = 154.21$ with 29 df, and for the 5.0-ft. circle $\chi^2 = 74.02$ with 29 df; both results are highly significant. In this instance, an aspect of the continuum theory is confirmed in which it differs from the discrete response theory. Numerous two-choice experiments have confirmed the matching

law that asymptotically the number of A_1 responses in any large block of trials is essentially equal to the number of E_1 reinforcements. Because of the smearing distribution in the continuum case, no such matching of response and reinforcement distributions is predicted, nor was it obtained.

Secondly, we tested for each circle the null hypothesis that the response distributions (Equation 9) with the estimated parameter a fit the response data. For the 2.5-ft. circle $\chi^2 = 47.56$ with 28 *df*, for which $P = .02$. For the 5.0-ft. circle $\chi^2 = 30.09$, also with 28 *df* for which $P > .30$. The somewhat disappointing result for the 2.5-ft. circle is entirely due to one cell which has an observed frequency of 66, whereas a frequency of 40.68 is predicted. The contribution of this one cell of the 30 to the χ^2 total is 16.15. Examination of the data for individual *S*s indicates that this large figure cannot really be explained by unusual behavior of one or two *S*s. The maximum contribution to the 66 total of any one *S* is 9. The fit of the remaining 29 cells is quite good, and, of course, the fit of the data from the 5.0-ft. circle is very satisfactory. The encouraging character of these results suggests a deeper analysis of sequential data in subsequent experiments; unfortunately, analytical computation of detailed sequential statistics is rather forbidding, although possible.

Summary

The study tests a formulation of stimulus sampling theory for a continuum of responses when reinforcement is determinate and the probability distribution of reinforcements is noncontingent, unimodal, and symmetric.

Sixty college students were run for 300 trials on a circular display apparatus. The *S*s' responses consisted of predicting the location of a spot of light on the circumference of a circle. At the end of each trial *S* was shown the "correct" location, which constituted the reinforcing event for that trial. The *S*s were divided into two groups, one of which used a circle with a diameter of 2.5 ft. and the other a circle with a diameter of 5.0 ft. The means, variances and third central moments of the last 200 responses for individual *S*s were tabulated. The results are in good qualitative agreement with the theory. In particular, the means matched closely that of the reinforcement distributions, but the variances were generally larger, as predicted by the theory, in contrast to the zero variance of the rational strategy of statistical decision theory.

The learning parameter θ and the smearing distribution which represents the conditioning of the single stimulus were estimated from the data. Asymptotic response densities were computed by use of the estimated smearing distributions and were found to be highly similar for both assumed forms of the smearing distribution and to agree well with the observed histograms for both groups.

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(Early publication received October 6, 1960)