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10 Rationality, habits and freedom

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10.1 Introduction

Our mental concept of ourselves is that of self-aware thinking beings. But contrary to much folklore psychology, we are almost entirely unaware or unconscious of our thinking processes. What we have excellent knowledge of is the results, not processes, of thinking. Empirical studies supporting this conclusion are surveyed in Section 10.2. Criticisms of expected-utility computations are made in Section 10.3. In terms of rational choice, deliberation or calculation is mainly reflected in constraints implemented as habits, which are considered in Section 10.4. Unconscious natural computations are fragmentary, occasional and contextual. They are in fact associations, as argued for in Section 10.5. The role of free associations is the subject of Section 10.6. The traditional characterization of freedom in terms of absence of constraints is reviewed in Section 10.7. Contrary to this long philosophical tradition of defining freedom as mainly the absence of constraints, I argue in Section 10.8 that uncertainty is an essential characteristic of freedom. It is uncertainly in elections, in markets and in competing cultural opportunities that is one of the most important features of free societies. Then in Section 10.9, I argue for entropy as the natural measure of freedom of individual choice, and of freedom of markets and elections. The use of entropy comes from ergodic theory. The basic isomorphism theorem of ergodic theory provides a direct way of comparing the freedom, in terms of uncertainty, of different elections and markets, or, if we wish, individual patterns of choice. Some substantive examples are given in Section 10.10. At the end, in Section 10.11, I sketch some additional measures of freedom that can add to what entropy contributes.

10.2 Unconscious nature of thinking

Our mental concept of ourselves is above all that of self-aware thinking beings. The pinnacle of rationality, and this conception of a person, is rational deliberation about ends, and means for achieving those ends. From Aristotle to the present, practical reasoning has been a focus of attention in philosophy, but in spite of the acuity of much of what has been written, the complexity and sophistication of the kinds of problems considered as presenting issues for the application of practical

reasoning have been limited. What has been especially missing has been attention to the large psychological literature on the nature of thinking, and in particular, the literature concerned with the thinking processes involved in making serious and seemingly deliberate choices that involve major personal goals.

Contrary to much folklore psychology and the implicit assumption of many philosophers, we are almost entirely unaware or unconscious of our detailed thinking processes. What we have excellent knowledge of is the results of thinking, often of partial results that constitute major steps in reaching a final decision about an important matter. Here is a relatively brief survey of the many kinds of experimental studies supporting these conclusions. They set scientific psychology in opposition to folklore psychology and numerous philosophical ideas and ideals about the rationality of practical reasoning. In fact, it is important not to imply a serious restriction to practical matters. The proper view of the unconscious nature of thinking processes applies to finding solutions to theoretical problems as well.

Two seminal articles on these matters are that of Nisbett and Wilson (1977), whose title is "Telling more than we can know: Verbal reports on mental processes" and Wilson (1985), whose title is "Strangers to ourselves: The origins and accuracy of beliefs about one's own mental states." These articles survey in depth a number of experimental and nonexperimental empirical studies over many years, including their own work. I give a brief summary here.

In the first category I mention studies concerned with the inability of individuals to answer "why" questions. Gaudet (1955) found that respondents could not explain why they liked particular political candidates. Ranging far afield from this, Kornhauser and Lazarsfeld (1955) found that respondents could equally not explain why they liked certain detergents for laundering purposes. Lazarsfeld (1931) found that respondents could not explain why they chose a particular occupation and, in a similar vein, Davis (1964) found respondents could not explain why they chose to go to graduate school. Further back in time, Burt (1925) found respondents could not explain why they became juvenile delinquents or, in terms of more positive decisions, Goode (1956) found respondents could not explain in any reasonable way why they got married or divorced. Rossi (1955) found respondents unable to explain why they moved to a new home.

In discussing these examples on several different occasions, I have chosen to expand upon the example of buying a new house. This is a traumatic and difficult process for nearly everyone who has been involved in it. Almost without exception, explanation of the particular choice made is woefully inadequate. This does not mean that certain constraints do not obtain. Individuals are quite competent to state constraints, such as location from schools, overall cost, age of the house and other such factors contributing in a significant way to the final decision. It is just that no overall rationale for the decision taken is ordinarily given. The usual reason is that most individuals, or families, who are selecting a new home, make a very wide search for candidates. They end up with a smaller list with the property that no one dominates all the rest. Consequently, the final decision is based upon something different from the application of a final, solid constraint or a detailed, explicit computation.

Another class of studies, oriented toward theoretical rather than practical problems, concerns individuals' reports on problem-solving processes. Ghilesin (1952) collected data on creative problem solving, as he put it, from Picasso to Poincaré. He emphasizes that production by a process of purely conscious calculation seems never to occur. A classic study of Maier (1931) on combining extension cords for electrification purposes on a ceiling shows how unconscious problem solvers usually are of their pursuit of a solution. In mathematics there is widespread recognition that theorem-proving of any difficulty depends upon imaginative leaps very similar to memory retrieval, but clearly computational in character. The key idea, just like that of retrieval of a memory, comes into consciousness with no trace at all of how it was arrived at. There are numerous famous anecdotes by scientists and mathematicians about this process. I shall not review them here, but almost everyone is aware of what Hadamard and Poincaré have claimed in this respect. I have never heard a serious mathematician deny that this important role of unconscious processes was not in fact always at work in obtaining any significant mathematical result. Here is a short famous quotation from Hadamard (1945).

> One phenomenon is certain and I can vouch for its absolute certainty: the sudden and immediate appearance of a solution at the moment of sudden awakening. On being very abruptly awakened by an external noise, a solution long searched for appeared to me at once without the slightest instant of reflection on my part – the fact was remarkable enough to have struck me unforgettably – and in a quite different direction from any of those which I had previously tried to follow.
>
> (Hadamard, p. 8)

The attempts to explain this lack of awareness have produced a large number of new experiments, hypotheses and theoretical analyses from psychologists. Let me just summarize some of the reasons given for why we are unaware of our unawareness. The first is a confusion of content and process. This is not a separation usually made in ordinary talk about decision making, why we have chosen a certain goal or adopted certain means for achieving a certain goal. Second, we have detailed private knowledge of ourselves that is obviously not accessible to anyone else. We can confuse this information with the processes of thinking, because these processes are naturally intertwined with the data that are more or less private to a given individual. For example, each of us knows private historical facts about his own thought and action that can affect his thinking processes. Moreover, an individual can tell you his focus of attention at any given moment, which is in itself something quite different from an account of his thinking processes, but is natural to confuse with those processes. Still another factor is private, intermittent awareness of various sensations. We can be aware of seeing a car in the distance or a person nearby missing a step. Recording these observed objects or events can be mistaken for thinking about them.

Perhaps most important, almost all of us are capable of describing coarse intermediate steps in complex problem solving. For example, the many steps taken in

buying a house, from surveying various neighborhoods, calling an agent, making an escrow deposit, closing the bank loan, to the final dramatic act of moving in. These intermediate steps are intermediate results, easily externally described, as opposed to descriptions of the associated thinking processes. But they are at the same time easily confused with the processes themselves, because we do not naturally separate our successive processes of thinking from our successive immediate results.

At the conclusion of this section what I want to emphasize is this: A theory of rationality that is posited on some exemplary style of rational deliberation, conscious, measured and complete, is utterly mistaken as a psychological account of how any of us go about making decisions about practical problems or solving theoretical ones.

10.3 Fantasies of expected utility computations

It is not just the philosophers of practical reasoning that have been mistaken, but it is also the economists and statisticians who have bought into the image of endless rational computations. The further the reach of the computations, the greater the sin of psychological omission in formulating the theoretical ideas. Perhaps the most excessive brand of this is Savage's (1954) famous fantasy of utility functions over possible future histories of the universe, and the related and intertwined fantasy of de Finetti that once we have a probability distribution, all future revisions of thought processes will be by conditioning only, that is, strictly in the sense of probability theory.

For reasons too numerous to enumerate here, the actual long-run calculations ever made are negligible. Keynes had it right. The important fact about the long run is that in the long run we are all dead.

The actual computations we do are fragmentary, occasional, contextual, driven by associations internal and external. A much better guide to thought than the utilitarian principle of maximization taken in its raw form is William James's account of the stream of thought in chapter IX of his *Principles of Psychology* (1890/1931). Here is one passage from James.

> Now we are seeing, now hearing; now reasoning, now willing; now recollecting, now expecting; now loving, now hating; and in a hundred other ways we know our minds to be alternately engaged. But all these are complex states.
>
> (James, p. 230)

Expanding upon this theme, I quote several additional passages, all taken from chapter IX. I quote them at such length here because I know of no place in the psychological literature that a better description has been given of the evanescent and fragmentary character of the stream of thought. In contemplating these quotations I do make the point that, regretfully, James is not clear about the distinction between being conscious and being unconscious of something. As you will see, these terms are not mentioned, but the absence of this distinction on his part,

important as it is to much of what I am saying, does not detract from the value of the wonderful, detailed descriptions of thinking he does give.

> Our earlier chapters have taught us to believe that, whilst we think, our brain changes, and that, like the aurora borealis, its whole internal equilibrium shifts with every pulse of change. The precise nature of the shifting at a given moment is a product of many factors. The accidental state of local nutrition or blood-supply may be among them. But just as one of them certainly is the influence of outward objects on the sense-organs during the moment, so is another certainly the very special susceptibility in which the organ has been left at that moment by all it has gone through in the past.
>
> (James, p. 234)

> But as the brain-tension shifts from one relative state of equilibrium to another, like the gyrations of a kaleidoscope, now rapid and now slow, is it likely that its faithful psychic concomitant is heavier-footed than itself, and that it cannot match each one of the organ's irradiations by a shifting inward iridescence of its own? But if it can do this, its inward iridescences must be infinite, for the brain-redistributions are in infinite variety.
>
> (James, p. 235)

> I am sure that this concrete and total manner of regarding the mind's changes is the only true manner, difficult as it may be to carry it out in detail. If anything seems obscure about it, it will grow clearer as we advance. Meanwhile, if it be true, it is certainly also true that no two 'ideas' are ever exactly the same, which is the proposition we started to prove.
>
> (James, p. 235)

> There is not a conjunction or a preposition, and hardly an adverbial phrase, syntactic form, or inflection of voice, in human speech, that does not express some shading or other of relation which we at some moment actually feel to exist between the larger objects of our thought. If we speak objectively, it is the real relations that appear revealed; if we speak subjectively, it is the stream of consciousness that matches each of them by an inward coloring of its own. In either case the relations are numberless, and no existing language is capable of doing justice to all their shades.
>
> (James, p. 245)

As I will argue shortly, our computations are built up from myriads of associations, intertwined with our past in ways that we can no more understand in detail now than we can explain how we retrieve a familiar name or a well-known fact from memory. It is why I like to say that when it comes to human computations, fragmentary and associative in character, Proust is a better guide than Turing. Here is a quotation that illustrates this well, from *Time Regained: In Search of Lost Time*, the last part of Proust's extraordinary novel (1927/1999).

> All day long, in that slightly too countrified house which seemed no more than a place for a rest between walks or during a sudden downpour, one of those houses in which all the sitting-rooms look like arbours and, on the wall-paper in the bedrooms, here the roses from the garden, there the birds from the trees outside join you and keep you company, isolated from the world – for it was old wall-paper on which every rose was so distinct that, had it been alive, you could have picked it, every bird you could have put in a cage and tamed, quite different from those grandiose bedroom decorations of today where, on a silver background, all the apple-trees of Normandy display their outlines in the Japanese style to hallucinate the hours you spend in bed – all day long I remained in my room which looked over the fine greenery of the park and the lilacs at the entrance, over the green leaves of the tall trees by the edge of the lake, sparkling in the sun, and the forest of Méséglise. Yet I looked at all this with pleasure only because I said to myself: "How nice to be able to see so much greenery from my bedroom window," until the moment when, in the vast verdant picture, I recognised, painted in a contrasting dark blue simply because it was further away, the steeple of Combray church. Not a representation of the steeple, but the steeple itself, which, putting in visible form a distance of miles and of years, had come, intruding its discordant tone into the midst of the luminous verdure – a tone so colourless that it seemed little more than a preliminary sketch – and engraved itself upon my window-pane. And if I left my room for a moment, I saw at the end of the corridor, in a little sitting-room which faced in another direction, what seemed to be a band of scarlet – for this room was hung with a plain silk, but a red one, ready to burst into flames if a ray of sun fell upon it.
>
> (Proust, pp. 9–10)

This long passage from Proust shows why he is a better guide to human computation than Turing. The true complexity of much, if not most, human computing is to be found in perception. The human visual system may be the most complicated system in the universe, after the brain itself. And our continual attention to vision, seen from an unusual angle, in Proust's highly particular perceptions and associations, is characteristic of much of our waking hours, even if we do not usually focus on what we see as intently as in Proust's account. This primacy of perception is testimony to the relative ease of building digital computers compared to the great difficulty of constructing artificial visual systems.

The gap between the richness and complexity of perception and thought, so well described by Proust and James, compared to the crude oversimplifications characteristic of any attempt at direct expected utility computations over future histories is an important source of skepticism about the latter.

10.4 Habits

There is a scent of *tabula rasa* about the approach to rational choice via maximizing expected utility. It is as if the organism has a simple, uncomplicated structure,

whose behavior can be maximized in the way that a simple physics problem can be solved by finding a maximum or minimum of an appropriate quantity. For biological organisms, beginning even with the simplest, nothing could be further from a sensible way of thinking about their behavior. The complexities that can be invoked at this point are much too numerous to be pursued in any detail, but there is one class of phenomena that may be seen not only in mankind but in animals up and down the hierarchy of evolution or complexity. If we were thinking about learning and if I were focusing on learning I would mention it. What are the effects of learning on the long-term behavior of an animal. There is, however, a better term, older, and also very much a part of folklore psychology, although not well developed. This is the concept of a *habit*. Some things that we call habits are undoubtedly purely instinctual, that is, are unlearned and encoded in the genes somewhere in the DNA. Most things, however, that we call habits represent an interaction between the genetic structure of an animal and the environment in which it develops and continues to exist. Habits are superb examples of learning, but I want to put the emphasis here on the results of learning, rather than on the learning itself.

Before I say more about habits, let me put my cards face up on the table, so that it will be clear how I am using the concept of habits as a help in characterizing rationality. Habits constitute restraints, in the standard mathematical sense of constraints, on the choices we make. We do not consciously think of our habits in making choices, but concentrate, so far as we exercise conscious discrimination at all, in choosing one thing rather than another, in such a way as to satisfy the appropriate constraints. For example, I am at the stage of my life where I very much prefer wine to beer. At an ordinary dinner in a restaurant, faced with a menu, I only think about the choice of wine, and almost never consider beer. I don't go through any deliberate, rational analysis of the virtues of wine over beer, because of the constraint already established by long-settled habits. I accept the constraint without even thinking or being conscious of it. I can, of course, at another time and for another purpose, make myself conscious of having this constraint. But the important point about habits is that in the act of choosing itself, we do not ordinarily pay conscious attention to the habits we have.

This is not to say that such conscious occasions can never occur. It is the stuff of family drama and the essence of many good novels for a person, real or fictitious, to face up to habits that must be broken, in order to make a choice that is much more important and meaningful to the person than any breaking of habits of old. But this is the exceptional situation – one that we can, of course, describe. Yet it is important to get the usual regime of choosing properly thought out. In fact, in the context of this article, I will not attempt to give a serious discussion of when we want to breach our constraints, that is, our habits, and go for something unusual, challenging or even frightening. This is an important topic, but one that can be left to the side, because of the low frequency of such choices, and the necessity of having a much better view of the usual kind of choices we make, from the dramatic ones of buying houses to the trivial ones of buying groceries.

So, I emphasize, the habits of a lifetime, as the saying goes, present constraints that are ordinarily satisfied. But the constraints do not fix the choice. My strong

constraint of always choosing wine, and never beer, does not in any way determine the particular choice of wine on a given occasion.

You may think that I am next going to say that we have come upon the proper role for maximization, namely, to maximize our choices subject to the constraints of habits. But I will not even accept the traditional theory of maximizing expected utility in this reduced role. To anticipate what I will say later, and to give you a sense of the organization of the ideas about rationality I am presenting, the next step after habits is to let the associations of the moment make the choice as freely and as easily as possible. I won't say more about these associations until the next section, but this is just a prelude to what is to replace, not just maximization, but even satisficing.

AQ: Is 'satisficing' spelling intended?

Now back to habits. Much of what I want to say in the context of the present article about habits is said better and in more detail in chapter IV of James's *Principles of Psychology*. I shall not attempt a faithful summary of James's ideas, but only emphasize points that are relevant to the characterization of rationality, and I do not claim that what I say is anything like a faithful paraphrase of James's thoughts. It is just that I have been much stimulated by reading his excellent analysis.

The first point is that habits are really physical and already present in nonanimate matter. What we ordinarily think of as certain material properties correspond to what we would call habits in animals. But particle or animal, the habit should be thought of as something physically embodied in the nervous system, and in the muscles, where appropriate. The only real difference on this score between animals and inanimate objects is the much greater mutability of habits in animals. James has a wonderful quote from someone else about the many ways in which matter itself is not immutable. The examples are particularly from designed objects, which have a special property. This is the second point: they function better the more they are used. Engines, locks, hinges on doors and the like improve with age, up to a point of course. Let me quote James (p. 112), "habit simplifies the movements required to achieve a given result, makes them more accurate and diminishes fatigue." The ironic thing about this aspect of habit is to recognize the importance of efficiency and yet to realize how little it is ever given its pride of place in the discussion of such matters by utilitarians. Habits, indeed, are themselves utilitarian in the deepest sense of that word, namely, in their clearly useful contribution to doing things.

The third property to be mentioned, one of importance in connection with mistaken notions of rational deliberation, is that habits diminish the conscious attention with which acts are performed. In more domains of experience than can be named, only the inept, the awkward and the untrained are conscious of their performances. The accomplished, the gifted and the well trained are not. And so it is with choices. The final process of choosing I have emphasized earlier on is one that is properly left unconscious, once the first round of constraints that are either habitual, or deliberately modified for application to a new situation, have been satisfied. The final reduced choice set should be one worthy of unconscious contemplation and free association. Now many will think that my phrase 'unconscious contemplation' is really overdoing it. Only the mindless choose this way. The data show otherwise.

Only the inept are mindful of their final choices, to put the matter in the most controversial way, but one about which I am all the same utterly serious.

Finally, I cannot forego one more quotation from James about the important social role of habits. This topic lies somewhat outside my main focus here, which is on individual rationality, but a theory of rationality that ignores the social framework, of one kind or another, in which all of us live, is a Robinson-Crusoe view that is clearly a reductive absurdity.

> Habit is thus the enormous fly-wheel of society, its most precious conservative agent. It alone is what keeps us all within the bounds of ordinance, and saves the children of fortune from the envious uprisings of the poor. It alone prevents the hardest and most repulsive walks of life from being deserted by those brought up to tread therein. It keeps the fisherman and the deck-hand at sea through the winter; it holds the miner in his darkness, and nails the countryman to his log-cabin and his lonely farm through all the months of snow; it protects us from invasion by the natives of the desert and the frozen zone. It dooms us all to fight out the battle of life upon the lines of our nurture or our early choice, and to make the best of a pursuit that disagrees, because there is no other for which we are fitted, and it is too late to begin again.
>
> (James, p. 121)

We don't have to accept or use all of James's examples. We can easily write new ones suitable for our own age and technology, but his point is understandable without any changes needed.

Formal definition of habits. Briefly, let $r(i, n)$ be the ith response on trial n. Let $T(n)$ be the presentation set of stimuli on trial n. I define a habit as a stochastic process in the responses $r(i, n)$ with constant set $T(n)$ of stimuli. In particular, assume to begin with, this process is, to first approximation, a first-order Markov process. In analyzing panel data on consumer purchases or any similar data set, it will also be important to test for second-order versus first-order dependency in the response data, and possibly further order dependency. By considering just a stochastic process in the responses, we ignore fluctations in the stimuli, which are important for the free associations discussed later.

For most data sets, I would expect the estimated first-order Markov chain to be ergodic, that is, it has a unique asymptotic mean distribution of responses independent of the initial distribution. For all such processes we can immediately compute the entropy rate, not the entropy of the cross-sectional distribution of response probabilities. These concepts are explained in more detail in Section 10.9.

A habit that is deterministic will, of course, have an entropy rate of zero. In my familiar example of usually choosing wine over beer in a restaurant, the entropy rate of my responses, at the concrete level of the kind of wine, vintage and winemaker selected, will not be zero. Notice that the level of abstraction selected will vary the entropy rate. It is also part of my philosophy of these matters that there is no ultimate concrete specification, so that any level selected reflects some kind of abstraction. As we eliminate vintage, say, first, then winemaker, and then kind of

wine, we expect the entropy rate to decrease, so that finally, if we have only the choice of beer, wine or soft drink, as the three possible choices, my entropy rate is close to zero. An interesting question for consumer-behavior studies is what level of abstraction is of the most interest in calculating entropy rate.

There is a deeper question and one I am not yet entirely clear about, but fundamental to the ideas I am working on about freedom. This is what is the proper level of abstraction, in terms of what is represented mentally (or in the brain). So, after making some determination of habit, if the entropy rate is not zero, room is left for free associations. It is especially the free associations that we expect to be malleable and therefore subject to transient changes in stimulation, the fundamental conviction also of firms that vie for shelf space to advertise their products.

My tentative answer to the level of abstraction of the associations is that it just depends on the strength of resemblance or similarity between the mental (or brain) images, on the one hand, and the stimuli on the other. And, in fact, it is a mistake of mine to introduce the misleading idea of abstraction. It is better to introduce different relations of similarity, which we can use to make corresponding, but more psychologically realistic, claims. In other words, from a psychological standpoint, any claim about abstraction should be backed up by a working concept of similarity or isomorphism used to define the particular level of abstraction.

Both the concept of habit and of free association can be applied with varying concepts of similarity or isomorphism. Note that the two must go together, if we want to complete the study of choice. For example, to use again my familiar example, if we consider just my standard choice of wine over beer, habit completely accounts for my choice at this level and there is no room left for free association. But if we make the isomorphic or similarity relation more detailed, there is. In fact, in the present formulation of ideas, whenever the level of characterization of a habit has nonzero entropy, the remaining nontrivial choice set leaves room for associations.

This remark leads to the natural question of how to distinguish between habit and free association. Can we just define a relation of isomorphism or similarity at any level and thereby mark a distinction, so that we distinguish only relative to such a relation? In some ways this seems a good choice, for after all, according to the ideas being advanced here, association or, the special case of association we label conditioning, is also at the basis of habit, except possibly for some small part that is genetic in character. Is such a complete relativization of the distinction between habit and free association a satisfactory answer? I don't think so. For, it seems to me, it is important also to separate the ephemeral quality of free associations from the lasting quality of habits. This separation can be made by introducing further distinctions among the similarity relations used, based on their temporal character. I do not pursue the formal details here.

10.5 Associations as natural computations

In spite of continued controversy about the nature of computations in the brain, there is a long tradition of giving pride of place to associations. The classical

philosophical text on association is to be found in David Hume's *Treatise of Human Nature* (1739/1988). Here is the following famous passage early on, near the beginning of the *Treatise*.

> As all simple ideas may be separated by the imagination, and may be united again in what form it pleases, nothing wou'd be more unaccountable than the operations of that faculty, were it not guided by some universal principles, which render it, in some measure, uniform with itself in all times and places. Were ideas entirely loose and unconnected, chance alone wou'd join them; and 'tis impossible the same simple ideas should fall regularly into complex ones (as they commonly do) without some bond of union among them, some associating quality, by which one idea naturally introduces another. . . . The qualities, from which this association arises, and by which the mind is after this manner convey'd from one idea to another, are three, *viz.* RESEMBLANCE, CONTIGUITY in time or place, and CAUSE and EFFECT.
>
> (Hume, pp. 10–11)

In spite of a long and powerful intellectual and scientific tradition supporting the dominant fact of association, it remains the case that trying to reduce the psychological processes of reasoning and choosing to those of association is a veritable bête noire for many psychologists and philosophers. I do not have the space to add my own fire to that wide-ranging battlefront, but I do note that with the rise of the neurosciences, once again the associative areas of the brain play a prominent role in scientific conception of how humans, especially, do so much reasoning and symbolic computing – note, of course, that the associative area is much larger in man than in any other mammal.

From a philosophical standpoint, the great opposition to the fundamental mechanisms of the mind being just associative computation and memory is the Kantian line of transcendental idealism grounded in the a priori synthetic. But it is important to note that Kant thought that Hume was right in what he claimed empirically for association (*Critique of Pure Reason* A100). It is just that he did not accept that Humean empiricism was ultimately enough as a foundation for science, especially for Newtonian mechanics and mathematics.

Writing a hundred years later, William James is an enthusiastic critic of Kant's grounding of science with necessary a priori synthetic principles. Here is a passage expressing his thought well.

> If pure thought runs all our trains, why should she run some so fast and some so slow, some through dull flats and some through gorgeous scenery, some to mountain-heights and jewelled mines, others through dismal swamps and darkness? – and run some off the track altogether, and into the wilderness of lunacy? Why do we spend years straining after a certain scientific or practical problem, but all in vain – thought refusing to evoke the solution we desire? And why, some day, walking in the street with our attention miles away from that quest, does the answer saunter into our minds as carelessly as if it had

> never been called for – suggested, possibly, by the flowers on the bonnet of the lady in front of us, or possibly by nothing that we can discover? If reason can give us relief then, why did she not do so earlier?
>
> (James, pp. 551–552)

The reader will notice that the phrase "pure thought," in the opening lines of this quotation, and the reference to reason in the closing sentence, refer, of course, to mistaken purely systematic conceptions of thinking. James goes on to make the important point that one of the great problems of giving too much of a role to reason, or necessary laws of nature, is how to account for variation or, especially, "errors" in behavior of man or matter. The problem here is amusingly reminiscent of the problem that theologians have had with explaining how God could permit the existence of evil in the world.

Moving ahead quickly to more recent developments in psychology, the special case of association that is important, in the first half of the twentieth century in the development of psychology, is of course conditioning. The concept of conditioning dominated thinking about almost all aspects of psychology from the first decade of the twentieth century to the second half of the century. It ended only with the linguistic revolution of Chomsky and others in the second half of the century, and the subsequent development of a cognitive psychology that, to a large extent, has emphasized the role of rules over associations as the basis for thought. This regime of cognitive psychology, which was prominent from about 1965 to 1980, has had, as its hallmark, the replacement of nonsymbolic by symbolic thought. The decline of this line of theory began around 1980 with the introduction of nonsymbolic computational processes, so characteristic of modern neural networks. Two centuries after the death of David Hume in 1776 we again find ourselves returning to associations, now often in the form of neural networks. Currently they occupy the dominant place in the conception of the mechanisms of thought. Not everyone will agree with the formulation I have just given. Many will claim that it is still just too strong to say this, that there are other modes of thinking that remain of great importance. I am skeptical of that. I am happy to push the thesis that those other modes are themselves splendid examples of conditioning, for example, the mental computations of arithmetic, the algorithmic rules we all learn early. If we turn from such algorithms with the contempt with which many cognitive scientists and some mathematicians do, then the response is even better. Surely the evidence is that the best and hardest mathematical proofs arise, not from some linear, nicely formulated line of explicit reasons, but from random, scattered, jumbled associations of the kind mentioned in the passage from James and the one from Hadamard. Only later is an orderly exposition of justification found.

To push these ideas further, in 1969 I gave a clear mathematical proof that, just from ideas of stimulus and response, we could generate finite automata (Suppes 1969). In a later article (Suppes 1989), I showed how to extend these ideas to an arbitrary Turing machine, all operating by conditioning, that is, by special cases of association. From a psychological standpoint these psychological constructions of finite automata or simulated Turing machines, are too simple. No doubt the

actual computational processes in the brain using associations extensively are more devious and complicated. Moreover, we do not begin language learning with a mind that is a tabula rasa. Much structure and related processing is constrained by our common genetic inheritance. It is then above all association or conditioning that shapes the further development. I will not develop further these ideas, because they are not really needed for what I want to say in greater detail about rationality and freedom. But it is important to emphasize the central role that association continues to play in thinking about brain processes.

There is one additional point I want to make to those who remain skeptical about association. Think about your own methods of memory retrieval, and then try to give me a theory that does not deeply involve processes of association.

10.6 Freedom of association

As some may note, the title of this section is meant as a double entendre. On the one hand, I have in mind associations in the brain, and on the other, the great historic libertarian demand of freedom of association for the individual. But it is the brain about which I am serious at this point. Let me be explicit about what I want to mean by *freedom of association*. I have in mind a hierarchical conception of how we make rational choices. To begin with, we must satisfy our habits. With satisfaction of the constraints given by habits we are then left with an unresolved set of choices. How should we choose from this set? The classical utilitarian method is by maximizing utility. The classic algebraic theory I consider a hopeless enterprise, for reasons already given. The rational individual, who is choosing gladly and happily, is one who is freely associating and choosing that one of the available set that seems most attractive, because of the depth of past associations that are brought up, as can be the case in buying a house, or, in other instances, by the association to anticipated events. Often, a glimpse at something attractive nearby sets off the train of associations. Belief in the relatively high frequency of this last case is a fundamental tenet of advertising.

The immediate reaction of some readers may be to challenge this probabilistic mechanism of choice as normal. They may recall (perhaps I should say, *associate*) their earlier encounter with the literature of psychoanalysis and its emphasis on the central role of free association in interpreting dreams or analyzing repressions, slips of the tongue and many other phenomena. But the central role of association in our mental life was not a Freudian discovery. It goes back at least to Aristotle. Here is Freud describing the associations arising from the interpretation of a dream:

> And next, we obtain these associations. What they bring us is of the most various kinds: memories from the day before, the 'dream-day', and from times long past, reflections, discussions, with arguments for and against, confessions and enquiries. Some of them the patient pours out; when he comes to others he is held up for a time. Most of them show a clear connection to some element of the dream; no wonder, since those elements were their starting-point.
>
> (Freud 1971, p. 11)

It does not sound much different from one of the earliest references to associations in various passages of Aristotle's *On Memory and Recollection*. For example,

> It often happens that one cannot recollect at the moment, but can do so by searching, and finds what he wants. This occurs by his initiating many impulses, until at last he initiates one such that it will lead to the object of his search. For remembering consists in the potential existence in the mind of the effective stimulus; and this, as has been said, in such a way that the subject is stimulated from himself, and from the stimuli which he contains within him. But one must secure a starting-point. This is why some people seem, in recollecting, to proceed from *loci*. The reason for this is that they pass rapidly from one step to the next; for instance from milk to white, from white to air, from air to damp; from which one remembers autumn, if this is the season that he is trying to recall. . . .
>
> If one is not moving along an old path, one's movement tends towards the more customary; for custom now takes the place of nature. Hence we remember quickly things which are often in our thoughts; for as in nature one thing follows another, so also in the actualization of these stimuli; and the frequency has the effect of nature.
>
> (Aristotle, pp. 303–307)

In the last part of this passage, especially with the reference to frequency, Aristotle is distinguishing between natural and customary associations. Earlier in the passage, when he mentions *loci* he is referring to the ancient "artificial" art of memory by associating, for example, people with given places. Ancient and medieval texts are full of a wonderful range of examples of such use of spatial places as an aid to memory. Aristotle does not use a Greek term for association, but it is implied in phrases such as "pass rapidly from one step to the next" or when he says slightly earlier than the quoted passage "Arts of recollection occur when one impulse naturally succeeds another" (p. 301). Finally a few lines later on the same page he describes what are sometimes called his three laws of association.

> This is why we follow the trail in order, starting in thought from the present, or some other concept, and from something similar or contrary to, or closely connected with, what we seek.
>
> (Aristotle, p. 301)

Here *similarity* is just like Hume's *resemblance*, and *closely connected with* like Hume's *contiguity*.

The maxims and heuristics of the ancient art of artificial memory were aimed at the facilitation of memory, but the associations used, often with an emphasis on vivid and striking images, are not far removed from those Freud encountered in the free associations of his patients. (For the history of the art of memory, see Yates, 1966.)

To make another point, I want to say something more explicit about what I mean by *free* associations, since the general theory of associations covers a large part of executing practical activities. In such activities the associations are not free, but conditioned in a fixed sequence to accomplish the task at hand. As the standard phrase goes, they have become automatic. Free associations are of a different sort, used in search of memory, for example, but here just because the automatic method is not working. Free associations are more characteristic during moments of meditation or reverie, but also as unexpected intrusions of images unrelated to the task at hand, prompted by any of a great variety of possibilities.

The message I am trumpeting is that of learning to recognize the guidance and the help we can get from such associations, or perhaps even more, from those that do not rise to consciousness, but that are expressed in action by our actual choices. We often describe such choices as instinctual, as "the one I liked but I can't say why," or as "the one that seemed familiar but I can't explain why."

Free associations are a mixed bag, some come with positive affect and some not. A good example of "not" is to be found in the early pages of Joyce's *Ulysses* (1934, pp. 7–11) as Stephen Daedalus ruminates about the death of his mother following Buck Mulligan's remark that he killed her by his stubborn refusal to kneel and pray at her bedside as she lay dying. Such inward-turning ruminations can interfere with the quality of associations and thus of choices. Experimental confirmation of this claim is to be found in Wilson and Schooler (1991) and related studies referred to there.

The variety of empirical studies that I would classify as relevant to the understanding of free associations is very large. But there are two broad, not quite orthogonal, classifications of the most importance. One is, the distinction between those having positive or negative affect, and the other is between being inward or outward directed. The connection between ruminative, negative-affect associations and psychological depression have been much studied. The detailed complex conclusions cannot be summarized here, but a good overview is to be found in Nolen-Hoeksema (1991).

Even though I am persuaded that the theory of rationality, or of freedom for that matter, in the fullest sense should include the psychological concepts and problems mentioned in the preceding paragraph, it is not feasible to go further here. I do think there has been far too much separation between the conceptual approaches to choice behavior of economists, on the one hand, and social or personality psychologists, on the other. Only in the empirical studies of consumer behavior have we as yet seen a substantial reduction of this separation.

A formal remark on utility. Even though I am, as already expressed, skeptical of the grander schemes of how expected utility is maximized in the choices of ideally rational persons, there is a natural connection between the probabilistic phenomena of free associations in choice and random utility models. Such models assume that a person chooses the outcome that has the largest momentary utility value at the time of choice, but fluctuations in utility follow some postulated probabilistic mechanism, which, in the present case, would be assumed to be the fluctuations of free associations.

In the following definition, let A be a finite set of choice outcomes and let $\mathbf{U}$ be a function defined on A such that for each a in A, $\mathbf{U}_a$ is a random variable. Let P be the probability measure of the joint distribution of the random variables $\mathbf{U}_a$, a in A. Then $(A, \mathbf{U}, P)$ is a *random utility model*. The probabilistic preference for a over b can then be expressed by the equation

$$p(a, b) = P(\mathbf{U}_a \geq \mathbf{U}_b)$$

We get various special models by making particular assumptions. For example, a *binary* random utility model $(A, \mathbf{U}, P)$ is one for which only the binary probabilities $p(a, b)$ are given. An *independent* random utility model is one in which the random variables $\mathbf{U}_a$, a in A, are independent (Luce and Suppes 1965; Suppes *et al*. 1989, chapter 17). Falmagne (1978) shows that a natural condition of non-negativity is necessary and sufficient for a finite set of choice probabilities closed under subsets of A to be represented by a random utility model. Useful applications of this framework have been made in the empirical study of consumer choices by further specialization to a logit model (Theil 1969; McFadden 1974; Domencich and McFadden 1975; Guadagni and Little 1983).

The individual choice probabilities p_i can be shown to have, for the multinomial logit model, the simple form

$$p_i = \frac{e^{U_i}}{\sum_j e^{U_j}}$$

I cannot refrain from noting that, by combining the kind of stimulus–response learning theory exemplified in Suppes (1969) and Luce's choice axiom (1959), I derived this form of equation, with the addition of a scaling constant k in an early paper on behavioristic foundations of utility (Suppes 1961).

AQ: Please check. Suppes 1961 not in ref list

10.7 But what is freedom?

There is little doubt that both the common-sense concept of freedom and many philosophical analyses of freedom concentrate on the characterization of freedom as the absence of constraints. Moreover, such constraints are almost immediately qualified, so as to exclude our normal habits as constraints on freedom, and it is not ordinarily considered a constraint on freedom that as agents our bodies must obey the laws of physics. The most familiar characterization of the constraints opposed to freedom is that they are imposed by another agent. A free action of an agent is one that is not compelled or directed by another agent. There is much that has been said about this notion of freedom as absence of constraints by other agents. Here, however, I shall only discuss rather briefly a few central topics. Regard what I have to say in this section as only a preliminary to returning in the next section to a more probabilistic conception, that leaves plenty of room for free associations.

A first issue is the problem of internal psychological constraints. Both in folk psychology and in the law, it is common to say that an individual did not freely commit a certain crime to which he has confessed, and therefore, is not guilty. The

reason given is that the individual was subject to overwhelmingly strong irrational compulsions. In some cases, a plea of insanity is upheld.

Another familiar argument is that freedom can be a proper part of folk psychology, but at a deeper level, the very idea of freedom is an illusion, because everything is causally determined. Perhaps the most famous philosopher to advocate these two positions together, that is, the one of freedom as the absence of constraints by other agents and the doctrine of causal determinacy, was Hume. Hume's agenda is in certain respects rather special. In his famous chapter on liberty and necessity in Part III, Book II of *A Treatise of Human Nature* (1739/1888), Hume wants to make the case for there being a science of the mind comparable to the science of nature, exemplified by the recent triumphs in physics, especially those of Newton. He readily admits that we cannot give a detailed explanation, from a scientific standpoint, of much mental phenomena, but he rightly makes the point this is also true of physical phenomena. So he makes the claim that there is just as much reason to believe in necessity in the case of mental phenomena as in the case of physical phenomena. His point is to deny any absolute concept of liberty or freedom. Everything, physical or mental, is causally determined, as we would formulate the concept today or, as he would put it, causally necessary.

Another great philosopher who held similar views was Immanuel Kant. Within the realm of experience, Kant had a variety of detailed arguments as to why we should view all phenomena in experience as governed by the laws of nature, by which, he meant the laws of physics considered in a broad way. In the Third Antinomy about causality in the *Critique of Pure Reason* (1781/1997), Kant asserts as the thesis of the antinomy that the idea of a determinant sequence of causes extending ever backward in time is absurd. Any causal sequence must begin with an event that is absolutely spontaneous (freedom in nature). He rejects, however, this argument in the Antithesis and accepts throughout as part of his philosophical doctrine the complete determinism, or, as he (and Hume) would say, the necessity of the laws of nature. There is great subtlety about Kant's argument. A case can certainly be maintained that his final decision in analyzing the antinomies, in particular the Third Antinomy, was to make the concept of determinate causation a regulative idea and to admit that a completely compelling argument for its constitutive character could not be given.

Kant is also famous for having two other concepts of freedom. First is the concept of transcendental freedom, which is outside experience, that is, outside the framework of time and space, and therefore outside the laws of physics. The other concept is that of practical freedom, which is in many respects in its philosophical roots like Hume's concept of freedom as absence from constraints.

Still another issue for the agent-constraint view of freedom is that of the extent to which other animals possess such freedom. There is certainly a long tradition, related to both moral and theological concepts, that admits no place for freedom in the behavior of animals, but, now this seems rather ridiculous from the standpoint of modern biological ideas of evolution. There remain, however, even within the biological framework, issues about freedom for animals, especially as we go down the phylogenetic scale. Do aplysia have freedom? As much as I would like to

pursue further arguments here, all I want to say at this point is that the conception of freedom as absence from constraint by other agents has something important and correct about it. It does not mean that it is a complete and satisfactory analysis in all respects.

10.8 Uncertainty as essential[1]

What is to be emphasized to begin with is that even the suggestion that uncertainty is central to the fact of freedom is missing in the classical philosophical analyses mentioned above, and in the main philosophical successors to Hume and Kant, such as John Stuart Mill in his famous essay *On Liberty* (1859/1991). This omission continues in the standard literature of this century. Throughout the rest of this article I try to show that this omission is mistaken, and that intuitive features of freedom in many economic, political and social settings implicitly take some form of uncertainty for granted.

But first, here is Hume's famous definition of liberty in *An Enquiry Concerning Human Understanding* (1777/1902).

> But to proceed in this reconciling project with regard to the question of liberty and necessity; the most contentious question of metaphysics, the most contentious science; it will not require many words to prove, that all mankind have ever agreed in the doctrine of liberty as well as in that of necessity, and that the whole dispute, in this respect also, has been hitherto merely verbal. For what is meant by liberty, when applied to voluntary actions? We cannot surely mean that actions have so little connexion with motives, inclinations, and circumstances, that one does not follow with a certain degree of uniformity from the other, and that one affords no inference by which we can conclude the existence of the other. For these are plain and acknowledged matters of fact. By liberty, then, we can only mean *a power of acting or not acting, according to the determinations of the will*; that is, if we choose to remain at rest, we may; if we choose to move, we also may. Now this hypothetical liberty is universally allowed to belong to every one who is not a prisoner and in chains. Here, then, is no subject of dispute.
>
> (Hume, *Enquiry*, p. 95)

What does Hume have to say about uncertainty or chance in liberty as he defines it? The next quotation shows why, in Hume's conception of liberty or freedom as absence of constraint, there is no place for uncertainty.

> It is universally allowed that nothing exists without a cause of its existence, and that chance, when strictly examined, is a mere negative word, and means not any real power which has anywhere a being in nature. But it is pretended that some causes are necessary, some not necessary. Here then is the advantage of definitions. Let any one *define* a cause, without comprehending, as a part of the definition, a *necessary connexion* with its effect; and let him

> show distinctly the origin of the idea, expressed by the definition; and I shall readily give up the whole controversy. But if the foregoing explication of the matter be received, this must be absolutely impracticable. Had not objects a regular conjunction with each other, we should never have entertained any notion of cause and effect; and this regular conjunction produces that inference of the understanding, which is the only connexion, that we can have any comprehension of.
>
> (Hume, *Enquiry*, pp. 95–96)

Much as I admire Hume, I think that this argument about chance and uncertainty is wrong. He confuses chance or probability with the absence of necessity, or evident necessity. In fact, one can very well hold to Hume's deterministic doctrine, for the moment anyway, and still strongly argue for a proper concept of probability, based just upon complexity and make the fundamental concept of probability then, a subjective rather than an objective one. The ins and outs of this move I will not examine in detail here, although there is something pertinent about it. The main point I want to make, however, is that we have a very good working concept of uncertainty in ordinary thought and in scientific thought as well, independent of whether or not necessity reigns everywhere. Of course, when I say that "necessity reigns everywhere," I have in mind that necessity, for Hume, is identical more or less with our modern concept of determinism. If we want to be metaphysical about determinism, then we need to be subjective about probability and if we are willing to admit uncertainty as an objective fact of nature, then we can, if we choose, be objective about probability. In either case, from any working scientific standpoint, as well as from ordinary concepts and talk of the world, we will use and hold to a concept of uncertainty and use it continually in our ordinary and scientific thinking. This is my major point about uncertainty. I will not accept a Humean argument that I should not use such a concept as a proper common sense or scientific concept of great usefulness.

So what I want to claim is that the aspect of freedom, not sufficiently noted in this rich philosophical literature I have been referring to, is giving a proper role to uncertainty. This regrettable absence of uncertainty is also characteristic of much of the formal literature in economics on opportunity sets and freedom of choice. Excellent examples are Pattanaik and Xu (1998) and Bavetta and del Seta (2001), which contain many further references. An exception is Arrow (1995). I comment on this further in Section 10.10.

There is much ordinary talk supporting my view. Consider someone who acts always with the greatest possible regularity in a given area of experience. We are very likely to say of that person that he is simply caught up in his compulsions, or his habits if we want to be more generous. But that in any case he is, as far as this area of his experience goes, not free to make a real choice. We have in mind a person who is not one of Hume's prisoners, who is very much subject to the will of someone else, but someone who in the obvious sense is not subject to external constraint, but just has an excessive regularity to his habits. We can also say there is no freedom, when the external situation does not permit any choice. So, for

example, it is natural to say, when you only have one theater in town that I am going to the theater, but I am only going to this one because I have no choice. And, further, it is acceptable and natural to say that I don't have any freedom in this matter.

In the same vein, Hume is certainly right about the following circumstances that we all encounter. We aren't free to make a new substantive choice, but we have made a promise and under our ordinary habits we are going to execute that promise. So if I am on an errand for a member of my family and someone says to me, "Stop that errand and go to a movie which will begin in the next fifteen minutes," I can naturally and easily reply, "Well, I'm sorry, but I am not free to do that. I made a commitment and I am going to keep it." So we all understand that there is no freedom in such situations. In many other cases, however, the evidence of uncertainty is very much present, as in actually choosing which movie to look at in a modern cable–television environment, where there may be as many as thirty movies at a given moment ready to begin. Here, I do indeed have freedom and, furthermore, it is natural to describe myself as being free, because I have not yet made a choice, and for someone watching me also to say, "It is uncertain what he is going to choose. He is still free to decide."

So, this kind of freedom, the freedom to choose in the face of uncertainty, at the moment when one wants to choose, naturally occurs and for the purposes of the argument here, it need not be settled whether this uncertainty is deep in nature or is rather subjective in character and is the probabilistic notion of uncertainty adhered to by subjectivists in the theory of probability, who do not make a claim about uncertainty being a fact of the external world.

Accepting then, this aspect of uncertainty as a central characteristic of freedom, either subjectively or externally, I now move on to the measure of freedom.

10.9 Entropy as the measure of freedom

First, some technical details. A stochastic process $\mathcal{X}$ is an indexed family $\{\mathbf{X_n}\}$ of random variables. The index, discrete or continuous, is usually interpreted as time, and so it will be here. For simplicity and without any real conceptual loss, I consider only the discrete case with $n = 1, 2, 3, \ldots$, although some remarks will concern the doubly infinite case, $n = \ldots -2, -1, 0, 1, 2, \ldots$. The usual assumption about the collection of joint probability distributions of any finite subsequences of the random variables being consistent is made.

The appropriate concept of entropy for a stochastic process $\mathcal{X}$ is that of *entropy rate* $H(\mathcal{X})$ defined as follows:

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{n} H(\mathbf{X_1}, \ldots, \mathbf{X_n})$$

provided the limit exists. (Notice that $H(\mathbf{X_1}, \ldots, \mathbf{X_n})$ is just the entropy of the first n random variables. We convert to a rate by dividing by n.)

A (discrete, finite) Bernoulli process is a stochastic process that is a sequence $\mathbf{X_1}, \mathbf{X_2}, \ldots$, or possibly a doubly infinite sequence, with the $\mathbf{X_n}$'s independent and

identically distributed random variables with a fixed finite range of values. It is easy to show that such a Bernoulli process $\mathcal{X}$ has entropy rate

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{H(\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n})}{n} = \frac{nH(\mathbf{X_1})}{n} = H(\mathbf{X_1})$$
$$= -\sum p_i \log p_i$$

We take the measure of freedom to be the entropy rate of the process.

Consider a market over time in which m individuals are sellers and n are buyers. At each period each buyer makes a purchase from exactly one seller. As before, the uniform probability distribution on the set of m^n possible transactions would define a discrete (and finite-valued) Bernoulli process, which would be for m^n possible transactions the stochastic process with maximum entropy rate and thus the one of this size with maximum freedom.

AQ: Is any word missing here?

I simplify the analysis at this point by considering only the sellers as the states of the market process. The probability of each of the m states, that is, sellers, represents the probability a random buyer will choose that seller at the given time. In application of these ideas to market data we would often need to estimate $p_{i,n}$ for seller i at the end of time period n by the relative proportion of the market seller i had for that period and make no attempt to identify the behavior of individual buyers. This asymmetry in the treatment of buyers and sellers is common in the analysis of markets and correspondingly, in the case of elections for candidates and voters. However, it is to be emphasized that this limited kind of data analysis is not at all satisfactory for a study of market processes over time, when the entropy rate depends on the transition data for individual buyers, as will become clear in the sequel. I note here that a sample path for a buyer is the sequence of states occupied by the buyer from one time period to another, with the state representing the seller with whom the buyer has a transaction. Although I do not do it here, for actual data analysis it would be desirable to introduce a state corresponding to a buyer not making a transaction in a given time period. There is little doubt that most sellers would shudder at the utter randomness of a Bernoulli market from one period to the next, as would most candidates at a sequence of elections with a corresponding Bernoulli character. Many firms would accept, even if not maximally satisfied, a market that is about evenly divided among a relatively small number of sellers, but would be aghast at the utter lack of customer loyalty as the buyers randomly shifted at each period from one seller to another.

The necessity of considering the time course of a market, and not just cross-section data, in measuring freedom can be well illustrated by a market with just three sellers. We can look at the three-state Markov market with the transition matrix

	1	2	3
1	$1-2\epsilon$	ϵ	ϵ
2	ϵ	$1-2\epsilon$	ϵ
3	ϵ	ϵ	$1-2\epsilon$

As $\epsilon \to 0$, the entropy approaches zero, but the cross-sectional distribution remains $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. I think it is intuitively obvious that a market or election with 100 percent loyalty, that is, with $\epsilon = 0$ in the above analysis, is not free. Sellers or candidates need make no effort to compete. This is why merely cross-section data can be misleading.

More generally, for a stationary process the entropy rate as defined above, it can be shown, is equal to the conditional entropy rate, defined as

$$H'(\mathcal{X}) = \lim_{n\to\infty} H(\mathbf{X_n} \mid \mathbf{X_{n-1}}, \ldots, \mathbf{X_1})$$

provided the limit exists, which it does for stationary processes. For a (first-order) stationary Markov process, as in the example,

$$\begin{aligned} H'(\mathcal{X}) &= \lim H(\mathbf{X_n} \mid \mathbf{X_{n-1}}, \ldots, \mathbf{X_1}) \\ &= H(\mathbf{X_2} \mid \mathbf{X_1}) \\ &= -\sum_x p(x) \sum_y p(y \mid x) \log p(y \mid x) \end{aligned}$$

and so it is easy to show for the Markov market example as defined above that as $\epsilon \to 0$, $H(\mathcal{X}) \to 0$. (Hereafter, I drop the distinction between H and H' in view of their equality for stationary processes.)

I now turn to the concept that is critical for making entropy rate the essential measure of the freedom of a market or election process. I add the word "process" to emphasize we are considering processes, not one-time cross-sections. The central question is this. How do two markets, or a market and an election, for that matter, compare in their intuitive sense of freedom if they have the same entropy, and contrariwise? As far as I know, this is not a question that has been much addressed in economics or political science. There have been several prior uses of entropy to measure the one-time cross-section distribution of a market, as part of a more general consideration of indices of concentration (Encaoua and Jacquemin 1980; Curry and George 1983; Tirole 1988, chapter 5; Foley 1994), but not of a market as a stochastic process. More importantly, entropy, as an invariant feature of certain structural properties of stationary stochastic markets, was first examined in Suppes (1996). The basis of application is that in many cases of conceptual interest, two stationary stochastic markets or elections will have the same entropy rate if and only if they are isomorphic in the measure-theoretic sense. It is this latter concept that needs to be formally defined.

Let us first begin with a standard probability space $(\Omega, \Im, P)$, where it is understood that $\Im$ is a σ-algebra of subsets of Ω and P is a σ-additive probability measure on $\Im$. We now consider a mapping T from Ω to Ω. We say that T is *measurable* if and only if whenever $A \in \Im$ then $T^{-1}A = \{\omega : T\omega \in A\} \in \Im$, and even more important, T is *measure preserving* if and only if $P(T^{-1}A) = P(A)$. T is *invertible* if the following three conditions hold: (i) T is $1-1$, (ii) $T\Omega = \Omega$, and (iii) if $A \in \Im$ then $TA = \{T\omega : \omega \in A\} \in \Im$. In the application we are interested in, each ω in Ω is a doubly infinite sequence and T is the *right-shift* such that if

for all $n, \omega_n = \omega'_{n+1}$ then $T(\omega) = \omega'$. Intuitively this property corresponds to stationarity of the process – a time shift does not affect the probability laws of the process, and we can then use T to describe orbits or sample paths in Ω.

We now characterize isomorphism of two probability spaces on each of which there is given a measure-preserving transformation, whose domain and range need only be subsets of measure one, to avoid uninteresting complications with sets of measure zero that are subsets of Ω or Ω'. Thus we say $(\Omega, \Im, P, T)$ is *isomorphic in the measure-theoretic sense* to $(\Omega', \Im', P', T')$ if and only if there exists a function φ: $\Omega_0 \to \Omega'_0$ where $\Omega_0 \in \Im$, $\Omega'_0 \in \Im'$, $P(\Omega_0) = P(\Omega'_0) = 1$, and φ satisfies the following conditions:

(i) φ is $1-1$,
(ii) If $A \subset \Omega_0$ and $A' = \varphi A$ then $A \in \Im$ iff $A' \in \Im'$, and if $A \in \Im$

$$P(A) = P'(A')$$

(iii) $T\Omega_0 \subseteq \Omega_0$ and $T'\Omega'_0 \subseteq \Omega'_0$,
(iv) For any ω in Ω_0

$$\varphi(T\omega) = T'\varphi(\omega)$$

I emphasize that the isomorphism in the measure-theoretic sense of two markets, two elections, or a market and an election seems at the right level of abstraction. The isomorphism expresses that the two structures have the same degree of uncertainty and thus the same structural freedom, even though they differ considerably in other characteristics. The fundamental point is that our conception of freedom needs to be at a rather high level of abstraction in order to be conceptually useful. It would be of little use if we ended up by making the freedom of each market or election *sui generis*, and thus not comparable to any other. What we should have is a methodology for comparing degrees of freedom. The isomorphism in a measure-theoretic sense of two stationary stochastic processes provides the important step of giving us a meaningful basis in terms of uncertainty for judging equivalence in freedom. Note why this is so. The φ function mapping one process into another is measure-preserving, so there is a structural isomorphism between corresponding events of the two processes such that they have the same probability. It is precisely the fact that the mapping carries events into events of the same probability that supports the claim that isomorphism in the measure-theoretic sense represents equivalence of uncertainty, and thus, of freedom of markets or elections.

On the other hand, it is equally important to note that isomorphism in the measure-theoretic sense of two stochastic markets only means isomorphism in the structure of uncertainty, as I have called it. Such isomorphism does not imply observational equivalence, nor would we want it to. For example, a Bernoulli market and a Markov market with strong dependence from one period to the next can be isomorphic in the measure-theoretic sense but easily distinguishable by a chi-square test for dependence. What we want to be able to say about these two markets is that they are equivalent in terms of freedom, but clearly different in other respects.

To show how recent fundamental results are about the relation between entropy rate and measure-theoretic isomorphism, I note that it was an open question in the 1950s whether the two finite-state discrete Bernoulli processes $B\left(\frac{1}{2}, \frac{1}{2}\right)$ and $B\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ are isomorphic. (The notation here should be clear; $B\left(\frac{1}{2}, \frac{1}{2}\right)$ means that the probability for the Bernoulli process with two outcomes on each trial is that for each trial the probability of one alternative is $\frac{1}{2}$ and of the other $\frac{1}{2}$.) The following theorem clarified the situation.

Theorem 1 (Kolmogorov 1958, 1959 and Sinai 1959) *If two finite-state, discrete Bernoulli or Markov processes have different entropies, then they are not isomorphic in the measure-theoretic sense.*

Then the question became whether or not entropy is a complete invariant for measure-theoretic isomorphism. The following theorem was proved a few years later by Ornstein.

Theorem 2 (Ornstein 1970) *If two finite-state, discrete Bernoulli processes have the same entropy rate then they are isomorphic in the measure-theoretic sense.*

This result was then soon easily extended.

Theorem 3 (Adler *et al*. 1972) *Any two irreducible, stationary, finite-state, discrete Markov processes are isomorphic in the measure-theoretic sense if and only if they have the same periodicity and the same entropy.*

We then obtain:

Corollary 1 *An irreducible, stationary, finite-state, discrete Markov process is isomorphic in the measure-theoretic sense to a finite-state, discrete Bernoulli process of the same entropy rate if and only if the Markov process is aperiodic.*

Given a stationary stochastic market or election, the case is a good one for accepting entropy rate as an appropriate measure of freedom. To take advantage of the intuitions and results of ergodic theory this rather drastic abstraction has been used, a practice not uncommon in economics, but not to be commended. It is a task for the future to modify the theoretical framework to make it more empirically realistic, but still able to deal with markets or elections as dynamic processes over an extended period of time, not just in terms of a single cross section. (What is critical is approximate stationarity, and fortunately this can be statistically evaluated for the finite sequence of time periods available, a matter discussed in the next section.)

10.10 Examples of freedom in markets

To illustrate more concretely how empirical analysis of entropy rate for market or election processes can be undertaken, even if the stationarity assumptions are only roughly approximated, I consider two market examples, drawn from Suppes (1996).

The first example uses transition data observed six times for 264 buyers choosing one of eight soft-drink brands (Bass 1974). The buyers serving as subjects in the experiment were required to select a 12-ounce can of soft drink four days a week for three weeks from among the eight brands shown in Table 10.1. All brands were available on six of the twelve days. The estimated probability transition matrix shown in Table 10.1 represents the average of the five transition samples from the six days of complete observations. The stationary probabilities, for choice of each brand, which are the choice probabilities that would hold asymptotically for the given transition data, are shown to the right in the column labeled p_∞. The next column to the right shows the entropy of each row, that is, $-\sum_j p_{ij} \log p_{ij}$. The total entropy rate for the market, which is just the p_i-weighted average of the row entropies, is 1.85, as shown at the bottom right of the table.

In Table 10.2 I show the transition data on US car purchases from the 1985 New Car Buyer Competitive Dynamics Survey of J.D. Powers and Associates. The data given here are from a data reduction used in McCarthy *et al.* (1992). The original data set consists of 30,142 automobile purchases in 1985, together with the record of the previous purchases of each buyer. McCarthy *et al.* (1992) drew a 25 percent random sample of 7523 observations, a small number of which were eliminated because of incomplete records. The first part shows the estimated probability transition matrix for the sample drawn. The second part shows the California subsample and the third the non-California subsample. As can be seen the purchases are divided into four categories: Chrysler, Ford, General Motors and Foreign, with obviously all purchases of foreign cars lumped together in the last category. The total entropy of the national sample is 1.84, that of the California submarket is 1.93, and that of the non-California submarket is 1.79.

Table 10.1 Transition matrix for soft-drink choices

$n/(n+1)$	*Coke*	*7-up*	*Tab*	*Like*	*Pepsi*	*Sprite*	*D-Pep*	*Fresca*	p_∞	*Entropy*
Coke	0.61	0.11	0.01	0.03	0.13	0.06	0.01	0.04	0.29	1.45
7-up	0.19	0.45	0.00	0.06	0.14	0.10	0.01	0.05	0.18	1.82
Tab	0.08	0.12	0.16	0.36	0.08	0.04	0.08	0.08	0.03	2.38
Like	0.09	0.15	0.09	0.15	0.24	0.04	0.13	0.11	0.06	2.55
Pepsi	0.18	0.13	0.01	0.03	0.51	0.07	0.03	0.04	0.23	1.68
Sprite	0.11	0.18	0.03	0.07	0.16	0.33	0.03	0.09	0.10	2.27
D-Pep	0.09	0.05	0.18	0.09	0.12	0.09	0.26	0.12	0.04	2.52
Fresca	0.22	0.09	0.05	0.11	0.15	0.11	0.07	0.20	0.07	2.37
Total entropy										1.85

Table 10.2 Transition matrix for automobile purchases

	Chrysler	*Ford*	*GM*	*Foreign*	p_∞	*Entropy*
National sample ($s = 7523$)						
Chrysler	0.27	0.19	0.38	0.16	0.27	1.92
Ford	0.28	0.30	0.32	0.10	0.18	1.89
GM	0.27	0.14	0.49	0.10	0.39	1.74
Foreign	0.27	0.10	0.26	0.37	0.16	1.88
Total entropy						1.84
California subsample						
Chrysler	0.31	0.22	0.27	0.20	0.29	1.98
Ford	0.20	0.33	0.27	0.20	0.19	1.97
GM	0.31	0.12	0.37	0.20	0.28	1.89
Foreign	0.31	0.12	0.19	0.38	0.24	1.88
Total entropy						1.93
Non-California subsample						
Chrysler	0.27	0.18	0.39	0.16	0.25	1.91
Ford	0.27	0.36	0.29	0.08	0.18	1.85
GM	0.23	0.13	0.55	0.09	0.43	1.66
Foreign	0.27	0.10	0.29	0.34	0.14	1.89
Total entropy						1.79

It may seem surprising that the eight-brand soft drink market, as sampled, has almost exactly the same entropy rate as the four-brand analysis of the US national automobile market, but the explanation seems obvious. Coke and Pepsi dominate the soft-drink market as sampled in a way that is not the case for the automobile market, and here when I speak of *dominate* I refer to the whole process of repeated choice and the consequent transition matrix. The diagonal entropy of 0.61 for Coke is not matched in magnitude by any other estimated probability in the four matrices of Table 10.2. The closeness of the measure of entropy rate, and as proposed here, the measure of freedom, of these markets has a sound theoretical basis in the measure-theoretic concept of isomorphism. The corresponding qualitative intuitive argument given above supports the same conclusion.

10.11 Is entropy enough?

From a structural standpoint the theorems on entropy as a complete invariant for the isomorphism of two Markov processes provides substantial justification for entropy as the single most important measure of freedom. But is entropy enough? I think not, for reasons I want to consider in this final section.

The first point is a general one. The concept of freedom is used in evaluating many kinds of individual and societal choices and processes having many different properties. It is unlikely that entropy alone will adequately measure all the properties relevant to freedom. This generalization is based on the wide scientific experience of many different scientific disciplines with many different processes.

Almost none, I would claim, are characterized by a single number. Already in the relatively simple case of classical particle mechanics, the motion of a particle is characterized at a given moment by its position, velocity and acceleration, as well as its time-independent mass, and the forces acting on the particle. Of course, these quantities are not all independent, but are related in the fundamental laws of mechanics. No detailed analogy to freedom is intended here, only the recognition of the several different quantities needed in mechanics, as a hint of more quantities being needed in the analysis of freedom. I conclude by giving some examples.

Measurement of Diversity. For a given person or group, let there be just m properties or characteristics that are relevant for a candidate in an election or product in a market. The following analysis is adapted from Suppes (1996). To keep things simple, let these properties be only qualitative ones – either they are or are not possessed by a candidate or product. No quantitative measure is admitted in the present version. Then for m relevant properties there are just 2^m possible types of candidates or products for the given situation. My second simplification is to ignore the reasonable claim that different relevant characteristics have different importance, and therefore weights should be introduced to measure relative importance. (I return to this matter of weights later.) Given these two limiting assumptions of having only qualitative properties and only equal weighting of importance, it is straightforward to change the definition of entropy to be for the proportions of types, not individuals, in a given election or product market. I give the revised definition only for a Bernoulli process, not for the entropy rate of a Markov process, but it is clear the new definition extends to other processes immediately. Let T be the set of types and $t = 2^m$, the number of types generated by m properties. Then the entropy $H(T)$ is defined in the expected way by summing over types:

$$H(T) = -\sum_{i=1}^{t} p_i \log p_i$$

The form of the definition looks, of course, exactly the same. But now it is for types not individuals.

A weighted model of types is more realistic, because of the nearly uniform agreement that relevant political or economic properties, characteristics or issues vary in importance. Candidates' views on foreign policy matter more for almost everybody than views on the budget for windmills. As much as differences in color matter in choice of a car, large differences in price matter more. Let w_i be the weight assigned to relevant type i by an individual or group, normalized so that $\sum w_i = 1$ with $w_i \geq 0$ for all i. We can then define an entropy-like quantity

$$\begin{aligned} \mathcal{W}(T) &= -\sum_{i=1}^{t} p_i \log p_i (t w_i) \\ &= H(T) - E(\log W) - \log t \end{aligned}$$

which has the property that for the uniform distribution of weights, that is $w_i = 1/t$, we have $\mathcal{W}(T) = H(T)$, since for the uniform distribution

$$\mathcal{W}(T) = -\sum_{i=1}^{t} p_i \log p_i \left(t \cdot \frac{1}{t}\right) = H(T)$$

I also want to stress some limitations of this move to weighted properties. In practice, we cannot accurately identify the relevant properties of a product or a candidate. The free associations of those choosing, buyers or voters, as described in Section 10.5, will inevitably be too varied, rich and, in many ways, inaccessibly private to describe in more than a generalized probabilistic manner. Nonetheless, psychologists studying consumer behavior will be able to identify directions of association useful to marketing or election managers. This rich experimental literature, which has seen explosive growth in the past several decades, is full of empirical subtleties about consumer behavior that have never been a focus of the standard economic or statistical literature on making decisions so as to maximize expected utility. Some examples are Dhar (1997), Drolet *et al*. (2000) and Edgell *et al*. (1996).

Individual freedom. The move from diversity to individuals is a natural one. From the perspective of an individual who weights all his subjective preference on properties that are missing from products or candidates, the entropy of the given market or election is zero, because the weighted frequency of the type he desires to choose is zero. He has no freedom, and he is repelled by talk of the market being competitive or the election free.

What about the individual at the other end of the scale? His weight is entirely on a set of properties represented by all the products available in the category of item he wants to purchase or by all the candidates in a forthcoming election. His state should be a much happier one. The entropy is zero, but he can just choose "randomly" since all his habitual constraints are satisfied. He can, in fact, give in entirely to his free or unconscious associations, which may be more decisive than he realizes in making his actual choices.

There is a good deal more to be said about such individual differences, but I conclude by restricting myself to one important point.

An essential element of a free society is freedom of speech and thought. The individual who despairs of finding products or candidates that satisfy him can express his views in many different forums and media, perhaps with some hope of changing the situation. But even if little external change seems likely, the very possibility of freely expressing his views, without constraint or punishment, is of great psychological importance in his developing and keeping a constructive stance toward his own limited choices, an essential aspect of rationality I could not develop in any detail here.

Notes

1 Some of the content of this section is taken from my article (Suppes 1997).

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