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PROBABILITY CONCEPTS IN QUANTUM MECHANICS*

PATRICK SUPPES**
Stanford University

The fundamental problem considered is that of the existence of a joint probability distribution for momentum and position at a given instant. The philosophical interest of this problem is that for the potential energy functions (or Hamiltonians) corresponding to many simple experimental situations, the joint "distribution" derived by the methods of Wigner and Moyal is not a genuine probability distribution at all. The implications of these results for the interpretation of the Heisenberg uncertainty principle are analyzed. The final section consists of some observations concerning the axiomatic foundations of quantum mechanics.

1. Introduction.

The fundamental problem considered is that of the existence of a joint probability distribution for momentum and position at a given instant. The philosophical interest of this problem is that for the potential energy functions (or Hamiltonians) corresponding to many simple experimental situations, the joint "distribution" derived by the methods of Wigner [1932] and Moyal [1949] is not a genuine probability distribution at all.

If this "distribution" is accepted as the most reasonable one possible, then we may infer a stronger result than the Heisenberg uncertainty principle, namely, not only are position and momentum not precisely measurable simultaneously, they are not simultaneously measurable at all.

To make the discussion as accessible and elementary as possible, the next section is devoted to the most relevant probability concepts. In Section 3 the general expression for the joint distribution of momentum and position is derived, and then computed, as an illustration, for the ground state and first excited state of a one-dimensional harmonic oscillator. The implications for the Heisenberg uncertainty principle of the results obtained are analyzed in Section 4. The final section (Sec. 5) consists of some observations concerning the axiomatic foundations of quantum mechanics.

2. Some Probability Concepts.

For subsequent discussion it will be desirable to have at hand several notions from general probability theory. For some of the definitions given below questions of continuity and differentiability can arise. To avoid these issues, which have no bearing on the central theme, adequate smoothness properties will be assumed in general arguments, and will be exhibited in all particular cases.

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To begin with, we assume as the starting point for probability considerations the underlying sample space. Points in this space represent possible experimental outcomes. For example, the set of all sequences of 1's and 0's is an appropriate sample space to represent the possible outcomes of flipping a coin an infinite number of times. Or, to be more finitistic, the set of 16 finite sequences of four terms, each term being 1 or 0 represents the set of possible outcomes of flipping a coin four times. In addition, we need a countably additive probability measure on an appropriately specified (Borel) field of sets. (In the finitistic cases, the probability measure may be defined for all subsets of the sample space.)

A random variable is then simply a (measurable) function defined on the sample space. Later on, we shall be considering the position and momentum of a particle as random variables. For the experiment of flipping a coin four times, a typical random variable would be the function h whose values are the number of 1's in any experimental outcome, i.e., any point of the sample space. Thus, $h(\langle 1, 1, 1, 1\rangle) = 4$, $h(\langle 0, 1, 1, 1\rangle) = h(\langle 1, 0, 1, 1\rangle) = h(\langle 1, 1, 0, 1\rangle) = h(\langle 1, 1, 1, 0\rangle) = 3$, etc. The practical justification of random variables should be apparent; they permit simplification of the structure of experimental outcomes in order to concentrate on the aspects of the experiment considered most significant.

Let Ξ be the sample space, let P be the probability measure on $B(\Xi)$, the given Borel field of subsets of Ξ, and let $\mathbf{X}$ be a (measurable) real-valued function defined on Ξ, i.e., let $\mathbf{X}$ be a random variable on Ξ. Then the *distribution function* F of $\mathbf{X}$ is defined for every real number x as follows:

$$F(x) = P\{\xi : \xi \in \Xi \ \& \ \mathbf{X}(\xi) \leq x\}.$$

It is easily shown that F is a monotonically increasing function such that

$$0 \leq F(x) \leq 1$$

$$F(-\infty) = 0$$

$$F(\infty) = 1.$$

The derivative of F, which we assume exists, is the *density* f. Put the other way round,

$$F(x) = \int_{-\infty}^{x} f(x)\, dx.$$

The expected value or mean $E(\mathbf{X})$ is defined as:

$$E(\mathbf{X}) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

Let $\bar{x}$ be the mean of $\mathbf{X}$. Then the variance of $\mathbf{X}$ is defined as:

$$\sigma_{\mathbf{X}}^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)\, dx.$$

We now turn to the characteristic function of F. Let F be the distribution function of a single random variable. Then the complex-valued function φ of the real variable t such that

$$\varphi(t) = E(e^{it\mathbf{x}}) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) \tag{1}$$

is the *characteristic function* of F.

Note that

(i) $\varphi(O) = 1$

(ii) $|\varphi(t)| \leq \int_{-\infty}^{\infty} dF(x) = 1$

(iii) $\varphi(-t) = \overline{\varphi(t)}$.

A distribution is uniquely determined by its characteristic function and conversely. Moreover, various properties of a distribution can be inferred from properties of its characteristic function. Granted that F has a derivative f which is the density function, we obtain at once from (1)

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx. \tag{2}$$

From (2) we conclude that f is the Fourier transform of φ, and thus we obtain at once by the standard result for Fourier transforms the important inversion:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi(t)\, dt. \tag{3}$$

The analogue of (3) for two random variables will give us the joint density function of position and momentum.

In fact, developments corresponding to the above for the joint distribution of two (or more) random variables proceed in the expected fashion. Let $\mathbf{X}$ and $\mathbf{Y}$ be two real-valued random variables defined on the sample space Ξ. Their joint distribution $F(x, y)$ is defined as follows :

$$F(x, y) = P\{\xi : \xi \in \Xi \;\&\; \mathbf{X}(\xi) \leq x \;\&\; \mathbf{Y}(\xi) \leq y\}.$$

The joint density function $f(x, y)$ is defined by:

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x\, \partial y}.$$

The marginal densities $f_1(x)$ and $f_2(y)$ are defined by:

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x, y)\, dx.$$

The two random variables $\mathbf{X}$ and $\mathbf{Y}$ are (*statistically*) *independent* if for all x and y

$$f(x, y) = f_1(x) f_2(y).$$

The means and variances of $\mathbf{X}$ and $\mathbf{Y}$ are defined as above, using now $f_1(x)$ for $\mathbf{X}$ and $f_2(y)$ for $\mathbf{Y}$. The covariance of $\mathbf{X}$ and $\mathbf{Y}$ is defined as

$$Cov(\mathbf{X}, \mathbf{Y}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})\,(y - \bar{y})\, f(x, y)\, dx\, dy$$

and the correlation coefficient $\rho_{\mathbf{XY}}$ as:

$$\rho_{\mathbf{XY}} = \frac{Cov(\mathbf{X}, \mathbf{Y})}{\sigma_{\mathbf{X}}\, \sigma_{\mathbf{Y}}}.$$

Note that if $\mathbf{X}$ and $\mathbf{Y}$ are independent then the covariance and correlation coefficient equal zero, but the converse is not necessarily true.

The characteristic function of the joint distribution function $F(x, y)$ is defined by

$$\varphi(t, u) = E(e^{it\mathbf{X}+iu\mathbf{Y}}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{itx+iuy}\, dF(x, y). \tag{4}$$

Analogously to (2), we have:

$$\varphi(t, u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{itx+iuy} f(x, y)\, dx\, dy. \tag{5}$$

Analogously to (3), we then have by the Fourier inversion theorem

$$f(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-itx-iuy}\, \varphi(t, u)\, dt\, du. \tag{6}$$

3. Joint Distribution of Momentum and Position.

Consider now the momentum and position random variables $\mathbf{P}$ and $\mathbf{Q}$. Following (4) the characteristic function $\varphi(t, u)$ is defined by

$$\varphi(u, v) = E(e^{iu\mathbf{P}+iv\mathbf{Q}}). \tag{7}$$

Throughout this paper we shall for simplicity consider only time-independent phenomena. Using the Hilbert space formulation, let (ψ, ψ) be the inner product of a state with itself. Following the usual formalism, the expectation $E(\mathbf{R})$ of an operator $\mathbf{R}$ when the quantum mechanical system is in state ψ is simply $(\psi, \mathbf{R}\psi)$. In view of (7) the characteristic function $\varphi\,(u, v)$ for the joint distribution of $\mathbf{P}$ and $\mathbf{Q}$ is given by:

$$\varphi(u, v) = (\psi, e^{i(up+vq)}\psi). \tag{8}$$

Corresponding to (6) we then have from (7) and (8) by Fourier inversion

$$f(p, q) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(up+vq)}\, (\psi, e^{i(up+vq)}\psi)\, du\, dv. \tag{9}$$

For canonically conjugate operators $\mathbf{P}$ and $\mathbf{Q}$, i.e., $\mathbf{PQ} - \mathbf{QP} = \hbar/i$, it may be shown that (8) simplifies to[1]

$$\varphi(u, v) = \int \psi^*(q - \tfrac{1}{2}\hbar u)\, e^{ivq}\, \psi(q + \tfrac{1}{2}\hbar u) dq$$

[1] Henceforth the range of integration is understood to be $(-\infty, \infty)$ and notation for it is omitted.

and whence by Fourier inversion

$$f(p, q) = \frac{1}{2\pi}\int \psi^*(q - \frac{1}{2}\hbar u)\, e^{-iup}\, \psi(q + \frac{1}{2}\hbar u)\, du. \tag{10}$$

As is well known in probability theory not every characteristic function determines a proper probability distribution, and this is indeed the difficulty with (10).

The expression given by (10) for the joint density was first proposed by Wigner [1932]. The derivations just sketched follow Moyal [1949]. It is natural to ask how "inevitable" is (10) in the framework of classical quantum mechanics. To the writer it seems to be by far the most obvious approach compatible with standard probability theory and the formalism of quantum mechanics. Different approaches are to be found in Dirac [1945] and Feynman [1948]. Dirac's theory leads to distributions which are complex-valued and thus cannot be interpreted as probabilities; formidable mathematical problems beset the Feynman pathintegral approach.

Before considering the Heisenberg uncertainty principle in the light of (10), it will be useful to consider two simple one-dimensional cases, for one of which—the harmonic oscillator in the ground state—$f(p, q)$ is a genuine density, and for the other—the oscillator in the first excited state—it is not.

Ground State.[2] The potential energy is given by

$$V(x) = \frac{1}{2} Kx^2,$$

and the time-independent wave equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2} Kx^2\, \psi(x) = E\, \psi(x).$$

The solution of this equation in terms of Hermite polynomials is familiar from the literature. In the lowest energy state H_0

$$\psi(x) = \left(\frac{\alpha}{\pi^{1/2}}\right)^{1/2} exp(-\frac{1}{2}\alpha^2 x^2), \tag{11}$$

where

$$\alpha^2 = \sqrt{\frac{Km}{\hbar^2}}.$$

Thus

$$|\psi(x)|^2 = \frac{\alpha}{\pi^{1/2}}\, e^{-\alpha^2 x^2} \tag{12}$$

which is a normal density with mean zero and variance $\sigma^2 = 1/2\alpha^2 = \hbar/2\sqrt{Km}$.

We now apply (10) and (11) to obtain the joint distribution of momentum

[2] I am indebted to Professor Sidney Drell and Mr. Raymond Willey for some helpful suggestions concerning these two examples.

and position. For convenience of calculation, we replace p by the propagation vector $k = p/\hbar$. We have at once:

$$f(k, x) = \frac{1}{2\pi}\int \psi^*(x - \frac{u}{2})\, e^{-iku}\, \psi(x + \frac{u}{2})\, du$$

$$= \frac{1}{2\pi}\left(\frac{\alpha}{\pi^{1/2}}\right)\int exp[\alpha^2(x^2 + (\frac{u}{2})^2)]\, e^{-iku}\, du$$

$$= \frac{1}{2\pi}\left(\frac{\alpha}{\pi^{1/2}}\right) e^{-\alpha^2x^2}\ \ \frac{\pi^{1/2}}{\alpha/2}\, exp\left[-\frac{k^2}{4(\alpha/2)^2}\right]$$

$$= \frac{1}{\pi}\, exp\left(-\alpha^2x^2 - \frac{k^2}{\alpha^2}\right).$$

Or, in terms of the momentum p, the joint density is

$$f(p, x) = \frac{1}{\hbar\pi}\, exp\left(-\alpha^2x^2 - \frac{p^2}{\hbar^2\alpha^2}\right). \tag{13}$$

Integrating out x in (13) we obtain for the marginal distribution of momentum

$$f_1(p) = \frac{1}{\alpha\hbar\pi^{1/2}}\, exp\left(-\frac{p^2}{\hbar^2\alpha^2}\right), \tag{14}$$

which is a normal density with mean zero and variance $\sigma_p^2 = \frac{\hbar^2\alpha^2}{2} = \frac{\hbar}{2}\sqrt{Km.}$

And integrating out p in (13) we obtain precisely (12), i.e.,

$$f_2(x) = \frac{\alpha}{\pi^{1/2}}\, e^{-\alpha^2x^2}. \tag{15}$$

We note at once that the Heisenberg uncertainty relation is satisfied by the product of the standard deviations σ_p and σ_x of the two marginal distributions (14) and (15), for

$$\sigma_p\sigma_x = \frac{\hbar\alpha}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}\alpha} = \frac{\hbar}{2}. \tag{16}$$

It is interesting to note that from the full marginal distributions we get the equality (16) as a stronger form than the usual inequality—for this special case of the harmonic oscillator in the lowest energy level. It may also be noted that the distributions of momentum and position are statistically independent, because $f_1(p)\, f_1(x) = f(p, x)$. *A fortiori*, the covariance and correlation coefficient are zero. This prediction of statistical independence is one which is not discussed in the usual treatments of the harmonic oscillator —in fact, I have not seen it anywhere. Consideration of this prediction illustrates one of the difficulties in analyzing the foundations of quantum mechanics. The difficulty is that it is so hard to come by an exact account of the experimental data. For example, for what classes of experiments have

standard χ^2 goodness of fit tests been applied to test the null hypothesis that the observed data fit the quantum mechanical marginal distributions for momentum and position? Even more to the point, for the experimental setups corresponding to a potential energy $V(x)$ from which we derive a genuine *joint* distribution is it possible to collect data on this joint distribution? Admittedly my own ignorance of the experimental literature is partly responsible for these questions, but it would seem that not only the philosophical discussions of quantum mechanics but the standard treatises as well provide inadequate answers.

First Excited State. In the first excited state, we have from the literature

$$\psi(x) = \left(\frac{4\alpha^3}{\pi^{1/2}}\right)^{1/2} x \exp(-\frac{1}{2}\alpha^2 x^2),$$

whence

$$|\psi(x)|^2 = \frac{4\alpha^3}{\sqrt{\pi}} x^2 e^{-\alpha^2 x^2}.$$

Applying now (10) and (11), and again replacing p by the propagation vector $k = p/\hbar$, we have:

$$(17) \qquad f(k, x) = (\frac{1}{2\pi}) \left(\frac{4\alpha^3}{\sqrt{\pi}}\right) \int (x^2 - (\frac{u}{2})^2) \exp[-\alpha^2(x^2 + (\frac{u}{2})^2)] e^{-iku} du.$$

Integrating (17) we obtain

$$(18) \qquad f(k, x) = \frac{4}{\pi}\left[\exp\left(-\alpha^2 x^2 - \frac{k^2}{\alpha^2}\right)\right]\left(\alpha^2 x^2 + \frac{k^2}{\alpha^2} - \frac{1}{2}\right),$$

and the function $f(k, x)$ is negative for those values of k and x such that

$$\alpha^2 x^2 + \frac{k^2}{\alpha^2} < \frac{1}{2},$$

which means that $f(k, x)$ is not a proper joint density.

A quite different example for which $f(p, x)$ is not a proper density is given in Rubin [1959].

4. Heisenberg Uncertainty Principle.

For the momentum and position random variables the Heisenberg uncertainty relation is the inequality

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

On the basis of the general results in the preceding section I want to urge that many of the interpretations given this inequality are mistaken.

The first thing to note is that this inequality for the product of the standard deviations of two random variables in itself tells us nothing about the process

of measuring the values of these random variables. Suppose, for example, we can measure the height and the weight of any human being (at a given time) with absolute accuracy. It will still be the case that for any reasonably-sized sub-population of humans the product of the standard deviations of height and weight will exceed some positive number. The Heisenberg relation is disturbing because we think of 'identically prepared' particles having the same momentum and position, as we do not in the case of the height and weight of humans. But the disturbance of classical ideas is not only this psychological one.

It is commonly said that the Heisenberg principle shows we cannot measure both momentum and position simultaneously with arbitrary precision. As we have already remarked, there is in fact a logical gap between the relationship itself and this measurement statement. This logical gap is not, however, the main problem. The real point is that the uncertainty relation does not represent a genuine statistical relationship at all, for there does not in general exist a joint probability distribution of the momentum and position random variables. The real claim to be made is that when a proper joint distribution of momentum and position does not exist, then these two properties are not simultaneously measurable at all.

The conclusion that momentum and position are not simultaneously measurable at all does not follow from the Heisenberg relation but from the more fundamental results about the absence of a genuine joint distribution. There is no underlying sample space which may be used to represent the simultaneous measurements, exact or inexact. It is in this sense that the Heisenberg inequality is not a genuine statistical relation. On the other hand, it may be admitted that in a weaker sense it is a real statistical relation. For particles prepared in a state described by a given Hamiltonian or potential energy function, it may be possible to measure the position of some and the momentum of others. And we would expect the product of the standard deviations of these two sets of measurements to be equal to or greater than $\hbar/2$.

The physically important fact, which is not equivalent to the uncertainty relation, is that the results of the successive measurement of two non-commuting observables like momentum and position depend on the order in which the measurements are performed. Let the pair of numbers (p, x) represent in a one-dimensional case the result of measuring the momentum p and then the position x, and let (x', p') represent for an 'identically prepared' particle the results of measuring the position x' and then the momentum p'. The important thing is that in general $p \neq p'$ and $x \neq x'$. Formally this relation for non-commuting canonically conjugate observables is expressed by the fundamental commutation rule

$$PQ - QP = \frac{\hbar}{i},$$

which, under the interpretation urged here, is more physically significant than the uncertainty relation $\sigma_p \sigma_x \geq \hbar/2$. Weyl [1931] clearly shows the

importance of the commutation rules in the basic structure of quantum mechanics.

Koopman [1957] is concerned to argue that quantum mechanics provides no real evidence for changing the foundations of probability. And with this I am in complete agreement, for the simple reason that there is, so far as I know, no substantial argument for making any such changes. The use of functions like the improper joint densities of the last section is for purposes of calculation. In no sense do they help make a case for changing the basic laws of probability. Far from it, rather it may be argued that we can use their very improperness as a key to inferring what is not possible to measure experimentally. Koopman goes on to claim that the inability to measure momentum and position simultaneously is not an unusual one. He says the following (p. 100):

> The standard example of *incompatibility* is in quantum mechanics, where α and β are statements concerning the same components of position and momentum. But it is possible to give a much more obvious example. Suppose that, as a result of a gamma-ray mutation, an altogether different and unique creature were produced from rat parents, and suppose that its resistance to lethal doses of certain poisons A and B is of interest. If α is a statement concerning the number of days before it dies when injected solely with A, while β is a similar statement regarding B, then α and β are clearly incompatible, for the simple reason that you can kill your creature with A alone or else with B alone, but not "with both alone." Therefore no logical combinations such as $\alpha\beta$, $\alpha + \beta$ have any meaning: the situation assumed in the definition of α is inconsistent with that needed to make β meaningful. Nor is the situation improved by restating α and β as implications (α being interpreted as that the A experiment *implies* that the creature survives so many days, etc.), simply because, in order to become experimental propositions, a single experiment ("trial") must be capable of telling whether α is true ("succeeds") or false; and similarly for β—and we are back where we were.

His rat example is well put, but it goes too far in reducing the incompatibility of two quantum mechanical statements to what is essentially ordinary incompatibility. In his example the conjunction α & β would ordinarily simply be regarded as false. The quantum mechanical situation is more complicated. We have two functions, the momentum and position random variables, describing two quantitative properties of some physical phenomenon—in realistic language, of some physical particle. We consider the functions to be defined for every instant of time. We then feel uneasy when we cannot talk about the values of these functions at the same time t. In a given experimental setup that we can appropriately describe by a Hamiltonian or potential energy function, it is possible to derive the marginal probability distributions of momentum and position.

Moreover, it seems possible with some slight modifications to talk about the joint distribution of the tendency of the two poisons A and B to kill the rats in a and b days respectively. Suppose we find, for example, that according to the amount C_i of a certain organic substance in the kidneys of the rats the number of days until death with injection of poison A is $a_i \pm 1$ and with injection of B is $b_i \pm 2$, with probability one. On the basis of observation

of the amount C_i, in the kidneys of a rat we could then assert that the probability of the rat having the property of dying in a_i days from poison A and the property of dying in b_i days from poison B is such and such. And by studying the distribution of the given organic substance in the kidneys of rats of a particular colony, we could go on to make unconditional probability statements about the rats in the colony having the property of dying in a days from poison A and b days from poison B. Because the effects of only one poison can be observed, the inference to the joint distribution depended upon discovering controlling factors like the organic substance in the kidney. In fact, the aim of the scientist is to investigate more and more thoroughly the chemistry of poisons in order to predict more and more accurately the effect of giving any particular one. It is, so far as I can see, completely meaningful to summarize the results of such investigations by joint probability statements. Note that given C_i, the probabilities of a_i and b_i are statistically independent, but without this information they are not.

Koopman asserts in the passage quoted above that the conjunction α & β is meaningless. What I am claiming is that by rephrasing α and β in terms of properties or dispositions we obtain new statements α' and β' whose conjunction is perfectly meaningful. The defense of this change is the same as the standard defense against a too narrow criterion of empirical or operational meaningfulness in evaluating scientific theories. It is, I take it, now a truism that not all scientific concepts or terms are directly observable, and *a fortiori* are not simultaneously observable.

The qualitative difference between the rat experiment and a quantum mechanical one is that in the quantum mechanical case we seem to be prevented even in principle from talking about the joint distribution of momentum and position.

5. Axiomatic Foundations.

I conclude with some general remarks about the axiomatic foundations of classical quantum mechanics. It is widely but mistakenly thought that von Neumann provided an exact axiomatization in his well-known book on quantum mechanics. He gives there exact axioms for Hilbert space, but he does not adjoin in exact fashion an axiomatic characterization of quantum mechanics. (By 'exact' I do not mean in logical symbolism but according to the standards of axiomatization in geometry or abstract algebra.) Perhaps the clearest axiomatization yet given is that of Mackey [1957].

Briefly speaking, Mackey proceeds in the following fashion for the time-independent case. Let θ be the set of observables and let S be the set of states; any structure on the sets θ and S is explicitly stated in the axioms. The function $p(A, \alpha, E)$ is defined whenever $A \in \theta$, $\alpha \in S$ and E is a Borel set of real numbers. Intuitively $p(A, \alpha, E)$ is the probability of measuring observable A in set E when the state of the system is α. The first axiom states in fact that for every A in θ and α in S, $p(A, \alpha, E)$ is a probability measure in the

argument E on the set of all real numbers. The second axiom guarantees uniqueness of observables with a given probability distribution, and similarly for states. It is a kind of extensionality axiom for observables and states.

If $p(A, \alpha, E) = p(A', \alpha, E)$ *for all* α *in* S *and Borel sets* E *then* $A = A'$, *and if* $p(A, \alpha, E) = p(A, \alpha', E)$ *for all* A in θ *and all Borel sets* E *then* $\alpha = \alpha'$. The remaining axioms are more technical and will not be given here. Properties as two-valued observables are defined, and a certain partial ordering in terms of probability distributions on properties is defined. The final and most powerful axiom is then the assertion that the set of all properties under the given ordering is isomorphic to the partially ordered set of all closed subspaces of a separable infinite dimensional complex Hilbert space.

It is important to point out that Mackey's axiomatization is not complete, for to get a system that is adequate to analyze detailed physical examples, further axioms are needed to set up the proper correspondence between observables and operators. To solve particular problems it is by no means sufficient to know only that such a one-one correspondence exists; the correspondence must be given constructively by additional axioms.

From the standpoint of probability theory, the present formalism of quantum mechanics in terms of operators on a complex Hilbert space is not a natural one. It is apparent that no one approaching the subject from the standpoint of stochastic processes would be in any direct way led to this formulation. Because most of the observable predictions of the theory are in fact probabilistic predictions, the question naturally arises of giving an axiomatic formulation in terms more congenial to the theory of stochastic processes. It should be clear from Section 3 that such a formulation is possible in terms of quasi-probability distributions, a term commonly used for the joint distribution of conjugate observables because of the possible negativity of the distribution. On the basis of Moyal [1949] and Baker [1958] such a formulation is not too difficult to give, although it is beyond the scope of the present paper to enter into the details, particularly since the discussion here has been restricted to quantum statics.

Moyal's paper demonstrates the conceptual advantages of this statistical approach in making detailed comparisons between classical and quantum statistical mechanics. From a more general philosophical standpoint it seems to me there are at least two reasons for interest in the quasi-probability distribution formulation. The first is the negative but therapeutic one of reducing the interminable discussion of the wave vs. particle interpretation of quantum mechanics. A statistical approach to the observable quantities is neutral as between these two interpretations. On another occasion I hope to recast in terms of the present statistical approach the many distinctions introduced by Reichenbach [1944] in his philosophical analysis of quantum mechanics.

The second reason for the philosophical interest of the statistical approach to quantum mechanics has already been mentioned. It provides a more natural place for probability concepts, thereby leading to a formulation in which it is easier to discuss general philosophical concepts like those of deter-

minism ans causality and to compare in a relatively direct way the role of these concepts in probabilistic theories of phenomena outside the quantum domain. I have tried to illustrate in Section 4 how such a formulation can clarify the meaning of Heisenberg's uncertainty principle. Consideration of quantum dynamics in terms of quasi-probability distributions is also helpful in clarifying the status of causality in quantum mechanics, but this is a task beyond the scope of the present paper.

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