Mathematical logic for the schools

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Statement of project goals

In general terms the aim of the program is to deepen and extend the mathematical experiences of the able elementary school child at the broadest level of mathematics, the level of methodology and the theory of proof. The approach is through a study of modern mathematical logic, in particular that portion of it which is concerned with the theory of logical inference or the theory of deduction.

In modern times logic has become a deep and broad subject. Only in recent years have systematic relations between logic and mathematics been established and a completely explicit theory of inference formulated which was adequate to deal with all the standard examples of deductive reasoning in mathematics and the empirical sciences.

The concept of axioms and the derivation of theorems from axioms is at the heart of all of modern mathematics. The purpose of this project is to introduce the academically gifted elementary school child to modern mathematics and mathematical methods at a level which is rigorous but simple enough in presentation and context to permit relatively easy comprehension.

Specifically, the objective of the project is to experiment in the teaching of mathematical logic to classes of academically talented fifth- and sixth-grade children. Thus it is considered a program in enrichment for the more able mathematics student and a supplement to the elementary school mathematics curriculum. Questions of particular interest involve the capacity of children of the fifth- and sixth-grade age level to do the kinds of deductive proof which are characteristic of modern mathematics, specific factors of difficulty, and the possible transfer of skills of analysis and correct reasoning to other subject-matter areas.

In the study of logic, the child may be introduced to a way of using language and ideas precisely. The emphasis is upon clarity and precision in thinking, upon rigor and consistency. The introduction of the concept of form and structure in language, and analysis of structure are integral parts of the child's study. Thus it is clear that while this project is considered a supplement to the program in mathematics in the elementary school, the application of the concepts and skills learned is not limited to this subject. The principles of logical inference are universally applied in every branch of systematic knowledge.

* *Editor's note.* This material is adopted from a report prepared by Professor Suppes.

Logic has always been considered a college-level subject. The belief has been that it is too abstract for the understanding of the elementary school student. Evidence is mounting that logic is not too abstract for the elementary school child of age 10, 11, or 12, but on the contrary, that this age may represent the most propitious time for the introduction of abstract concepts.

The importance of the theory of proof and of the methodology of deriving theorems from axioms in modern mathematics cannot be questioned. Yet, development of the skills of deductive reasoning has been left largely to incidental learning in the school mathematics curriculum. The point of view represented in this program is that deliberate and well-planned teaching of mathematical logic will enlarge the scope of the able student of mathematics in the elementary school, and will provide a background for a deeper and more penetrating study of mathematics.

Pilot study: 1960-61

A pilot study in teaching mathematical logic to a group of selected fifth-grade students from a public school near Stanford University was carried on during the academic year 1960–61. The class consisted of twenty-five students who were selected from the several fifth-grade classes in the school on the basis of their arithmetic achievement scores. The children were given the arithmetic achievement test during the seventh month of the fourth-grade year. All students selected for the logic programs scored at the sixth-grade level or above. The class comprised approximately 27 percent of the total fifth grade in the school.

The class met three or four days per week throughout the school year for periods of about thirty or thirty-five minutes each. It was taught by Dr. Shirley Hill, a member of the Stanford Project staff. Textbook material, composed of explanatory material and practice exercises, was written for and used by the class. Experimentation with this material and subsequent revisions resulted in a textbook in mathematical logic for the schools. This book and its contents will be discussed in detail later in this report. The content covered by the pilot class is that which is contained in the first five chapters of the textbook.

Achievement tests given periodically indicated that this material is well within the capacity of the able fifth-grade student. Although the class proceeded much more slowly than the usual undergraduate class in college mathematical logic, the results indicate that children of this age are able to achieve a level of accomplishment comparable to college students in the skills of deduction essential to mathematical reasoning when instruction is geared to their level of experience.

Experimental program: 1961-62

Eleven experimental classes of selected fifth-grade students began the work in mathematical logic in September, 1961. These classes represented eleven different schools in six school districts in the San Francisco Bay Area. In each school approximately 25 to 30 percent of the fifth graders were selected by the administrators on the basis of general ability and high achievement in mathematics.

Each of the classes met three days a week for a period of between thirty and forty minutes. The groups were taught in each case by a classroom teacher from the staff of the school. During the summer of 1961, these eleven teachers completed a four-week intensive course in logic at Stanford University. The course was an introduction to mathematical logic with special emphasis on problems of teaching the subject matter to elementary school students. The book, *Mathematical Logic for the Schools*, was used in the summer course and was distributed to all experimental classes.

The organization of the work in the experimental classes followed the sequence

presented in the textbook. There was, however, no effort to maintain a uniform rate of progress in all classes. The teachers were not committed to covering any specific body of content during the first year, but were encouraged to proceed at the pace they felt to be most appropriate for their classes.

For the academic year 1962–63, it is planned to continue the program in mathematical logic with this same experimental group at the sixth-grade level. In addition, a new group of fifth-grade experimental classes will begin the program. The plan is to continue to train teachers for the experimental classes during both 1961–62 and 1962–63. During the school year 1961–62, the original pilot class continued in the sixth-grade portion of the program as a pilot group. Additional materials were developed for this class, and this experimentation will lead to the second book in the series, *Mathematical Logic for the Schools*. Book 1 is a self-contained body of content, but at this point it is impossible to ascertain whether the experimental classes will be able to cover the entire book in a single year. The sixth-grade classes will simply carry on where the fifth-grade classes end.

In order to evaluate the achievement level of the experimental classes, a series of tests will be given to the experimental group and results compared with those from the same test given to the regular undergraduate college classes in mathematical logic at Stanford University. These tests will be examinations of achievement in the subject matter of mathematical logic. Evidence from the pilot class indicates that its level of accomplishment was comparable to that of a a college class in mathematical logic, although its pace was much slower. It is suggested that the two groups, experimental and control, may be considered comparable in that both represent a selected group in terms of the general population. It is understood that this comparison of achievement will not be at comparable time periods but at comparable points of accomplishment in covering the systematically organized body of course content.

In addition to the achievement tests in the subject matter itself, general achievement tests will be given at the end of the first and second years. The purpose of the general achievement tests is to test for transfer and generalization, and these will be given to the experimental group and a matched control group of the same age and general intelligence from the same or similar school districts.

Textbook

The textbook used by all classes is *Mathematical Logic for the Schools* by Patrick Suppes and Shirley A. Hill. It contains explanatory material with an emphasis on exercises and examples. The organization of the book and its content is given below.

Part I: *Sentential Theory of Inference*

Chapter 1: *Symbolizing Sentences*—analysis of compound sentences; sentential connectives; symbolization.

Chapter 2: *Logical Inference*—rules of sentential inference; proofs, including conditional and indirect proofs; validity and invalidity; sentential interpretations; consistency and inconsistency.

Chapter 3: *Truth Tables and Tautologies*—truth-functional analysis; tautological implications and equivalences.

Part II: *Logic of Quantification*

Chapter 4: *Terms, Predicates, and Universal Quantifiers*—analysis of atomic sentences; symbolization of terms and predicates; atomic formulas; universal quantification.

Chapter 5: *Universal Specification and Law of Identity*—specification; derivations; logic of identity.

Chapter 6: *A Simple Mathematical System: Axioms for Addition*—derivation of theorems from axioms and definitions; commutative axiom for addition; associa-

tive axiom for addition; axiom for zero; axiom for negative numbers.

Chapter 7: *Universal Generalization*—derivation of general theorems with variables.

Teacher training

The teachers for the experimental classes received a four-week intensive course in mathematical logic at Stanford University. For those teachers who continued into the sixth-grade portion of the program an additional course in mathematical logic was offered in 1962. At that time another course was offered for teachers who wished to begin the fifth-grade program.

Close contact is maintained between the members of the project staff and the teachers of the experimental classes. In order to achieve this objective, monthly meetings are held with the group, and a member of the staff consults with the individual teachers periodically.

Fizzion of fractions

Many arithmetic texts and consequently many teachers have failed to emphasize and make use of the meanings of the words for the terms of a fraction.

I have not been able to determine who first applied the words "numerator" and "denominator." Whoever it was, that person understood functions, for these names are full of logical meaning.

The word "numerator" is related to the word "enumerate" which means to count. The numerator then is the *counter*. It tells how many we have of a particular class of objects. The names of these objects are given by the denominator. This word means to name, so the denominator could well be called the *namer*. Thus the terms of a fraction become the "counter" and "namer." The numerator counts what the denominator names.

All the natural numbers can be considered to be fractions with the namer being "one."

Fractions have the property that the "namer" can be changed without changing the value of the fraction if we make the related necessary change in the "counter." This property applies to the natural numbers if the namer "one" is used As far as numerical value is concerned, two of one class of objects may equal four of another class of objects. Thus 2 "thirds" equal 4 "sixths."

This property is that which is stated in the theorem that if both numerator and denominator are multiplied (or divided) by the same number, the value of the fraction is unchanged. It may seem odd to the student that we can, so to speak, multiply the name of the denominator by a number to get a new name.

A student may readily see that apples cannot be added to pears unless we change the names of both to the common name "fruits." Three apples and two pears are five fruits but not five hyphenated "apple-pears."

To add $\frac{1}{3}$ to $\frac{3}{4}$ or, stated verbally, to add one of the objects named "thirds" to three of the objects named "fourths," changes to a common name (class) are required. Let this common name be a "fizz." The 1 "third" can be changed into 4 "fizzes" and the 3 "fourths" into 9 fizzes. The sum is 13 fizzes.

It might be well to write these fractions in the form of 1 inch plus 3 yards or 1 third + 3 fourths = 4 fizzes + 9 fizzes = 13 fizzes. Of course, the evaluation of the sum is not complete until the numerical name "twelfths" for these particular fizzes has been determined.

To show application to division, $\frac{1}{3} \div \frac{1}{4}$ can be transformed into 4 fizzes $\div$ 3 fizzes. This can be interpreted as "4 fizzes can be divided into how many sets of 3 fizzes?" Compare 4 units $\div$ 3 units if some student asks how the "namer" can be "3 fizzes."

The application of fizzes to multiplication is not very useful. Examples:

$\frac{1}{3} \times \frac{1}{4} = \frac{1}{4}$ of 3 fizzes or 1 fizz
$\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$ of 3 fizzes or 1 fizz
$\frac{2}{3} \times \frac{1}{4} = \frac{2}{3}$ of 3 fizzes or 2 fizzes
$2 \times \frac{1}{4} = 2$ times one of the objects named "fourths" or 2 "fourths."

However, the above examples show a rule for multiplication of two fractions: change the second fraction to fizzes such that the number of fizzes is a multiple of the denominator of the first fraction.

If both fractions in the product $\frac{1}{3} \times \frac{1}{4}$ are changed to fizzes, we would then have 4 fizzes $\times$3 fizzes. This is not helpful for understanding multiplication of fractions. We do not think of 4 "units" $\times$3 "units" as equal to 4$\times$3 "units squared"; or of 4 "fifths" $\times$3 "fifths" as equal to 4$\times$3 "fifths squared." But we do think of 4 yards $\times$3 yards as equal to 4$\times$3 square yards.

The use of an unusual term as "fizzes" will emphasize the meaning of the numerator and denominator with respect to operations upon them.

In view of the current interest in nuclear reaction, the above procedure is called "fizzion of fractions."