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LOGICS APPROPRIATE TO EMPIRICAL THEORIES

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1. Introduction. To those like myself who are mainly concerned with the methodology of the empirical sciences, the present symposium is both sobering and encouraging. It is sobering as one thinks of the scientific contrast between the majority of papers read here and the standard sources in the methodology and philosophy of science. Yet it is encouraging, because the hope is engendered that many of the methods, and perhaps above all, the intellectual standards of these papers, will extend themselves in a natural way to logical investigations of the empirical sciences. The logical and philosophical foundations of physics, for example, seem to be at about the stage where the foundations of mathematics were during most of the nineteenth century. Nearly any physicist and a large number of philosophers are prepared to deliver at a moment's notice a lecture on the foundations of quantum mechanics. The situation is far different with respect to the foundations of mathematics. With an ever-increasing volume of deep and rigorous results, mathematicians unacquainted with the literature are not prone to deliver casually-put-together *obiter dicta* on logic and related topics.

The present paper is meant to be a quite modest contribution to the foundations of physics conceived of as the same sort of discipline as the foundations of mathematics. My objective is to give two sorts of examples of empirical theories for which it is natural and convenient to introduce a logic that is deviant in some respect. The first class of examples is derived from theories of measurement and the more general consideration of physical principles of invariance. The second class is drawn from quantum mechanics.

2. Principles of invariance and three-valued logic. In theories of measurement or in more general physical theories principles of invariance arise in a natural way. For example, the statement that Venus has greater mass than Earth is true or false, independent of what unit of

mass is used. Put another way, the truth value of the statement is invariant under an arbitrary (positive) similarity transformation of the units of mass used in attempting to verify or falsify the statement. On the other hand, the statement that the mass of Venus is greater than 10^{10} has no such invariance, for its truth or falsity will vary with the units of mass selected.

Let us generalize on this example by looking at a language in which we can express the results of n distinct measurements on physical objects. To avoid many repetitious details, I shall use as a basis the formal language of Tarski's monograph [51] enriched by individual variables "a", "b", "c", ..., which take as values physical objects and n function symbols "m_1", ..., "m_n". Terms are constructed from these function symbols and individual variables in the expected fashion; in particular, it is intended that $m_i(a)$ be a real number for $1 \leqslant i \leqslant n$. Let us call this language for a fixed n, $\mathscr{L}_n$. This formalization of $\mathscr{L}_n$ is necessary only to give definiteness to some of the logical results and problems it seems of interest to formulate. With inessential modifications for what is to be stated in this paper, $\mathscr{L}_n$ could easily be adapted to express in canonical form the results of a wide class of physical experiments.

The general semantical notion of a model of a language I shall assume is familiar. By a *model* of $\mathscr{L}_n$, I mean an ordered $(n+2)$-tuple $\mathfrak{M} = \langle \mathfrak{S}, A, m_1, \ldots, m_n \rangle$, where $\mathfrak{S}$ is the usual system of real numbers under the operations of addition and multiplication and the relation less than (technical details about $\mathfrak{S}$ are not critical here), A is a finite, non-empty set, and $m_1, \ldots, m_n$ are real-valued functions on A. The intended interpretation should be obvious. The set A is meant to be a set of physical objects whose various properties are measured by the numerical functions $m_1, \ldots, m_n$.

Associated with each function m_i expressing the results of a particular kind of measurement is a group G_i of (numerical) transformations. The group G_i characterizes the degree of uniqueness of the measurement. If m_i is a mass measurement, then G_i is the group of similarity transformations. If m_i is a classical position measurement on the x-axis, then G_i is the group of linear transformations, and so forth.

We first define when two models of $\mathscr{L}_n$ are appropriately related by these groups of transformations.

Definition 1. *Let* $\mathfrak{M}_1 = \langle \mathfrak{S}, A_1, m_1, \ldots, m_n \rangle$ *and* $\mathfrak{M}_2 = \langle \mathfrak{S}, A_2, m_1', \ldots$

..., $m_n'\rangle$ be two models of $\mathscr{L}_n$, and let $\mathfrak{G}=\langle G_1, \ldots, G_n\rangle$ be an n-tuple of groups of transformations. Then $\mathfrak{M}_1$ is $\mathfrak{G}$-related to $\mathfrak{M}_2$ *if and only if*

(i) $A_1=A_2$;

(ii) *for $1\leqslant i\leqslant n$, there is a φ in $\mathfrak{G}_i$ such that*

$$\varphi \circ m_i = m_i',$$

where $\circ$ is functional composition.

The intuitive idea of truth-value invariance under the appropriate groups of transformations is formalized in the next definition.

Definition 2. *An atomic formula F of $\mathscr{L}_n$ is* $\mathfrak{G}$-empirically meaningful *if and only if F is satisfied in a model $\mathfrak{M}$ of $\mathscr{L}_n$ when and only when it is satisfied in every model $\mathfrak{G}$-related to $\mathfrak{M}$.*

In other words, an atomic formula is said to be empirically meaningful only when it has the appropriate invariance properties—that the condition expressed in Definition 2 is always sufficient and not merely necessary is not a point of argument here. For a more extensive discussion of these matters see Suppes [59].

Definition 3. *A formula of $\mathscr{L}_n$ is* $\mathfrak{G}$-empirically meaningful *if and only if each of its atomic formulas is such.*

The reason for requiring that each atomic formula of a formula be meaningful is to obtain truth-functionality for the three-valued logic characterized in the next theorem. The substance of this logic is to assign the ordinary values truth and falsity to meaningful formulas, and the single value *meaninglessness* (abbreviated μ) to the remainder. The proof of the theorem is straightforward and is omitted.

Theorem 1. *Assign the truth value μ to $\mathfrak{G}$-empirically meaningless formulas of $\mathscr{L}_n$. Then the formulas of $\mathscr{L}_n$ are truth-functional in the three-valued logic T, F and μ defined by the following tables for negation and conjunction.*

S	$\neg S$
T	F
F	T
μ	μ

$\wedge$	T	F	μ
T	T	F	μ
F	F	F	μ
μ	μ	μ	μ

The ordinary definitions of disjunction, implication and so forth in terms of negation and conjunction are assumed in working out additional tables. The resulting truth tables are precisely those which Kleene [52a, p. 334] has called the *weak tables*, and they do seem to be about the weakest possible extension of classical two-valued logic.

A simple example will show why meaningfulness needs to be initially defined in terms of the atomic formulas rather than arbitrary formulas in order to have the truth-functional result of Theorem 1. Suppose we had proceeded the other way, and let m_1 be a mass measurement function with the associated group of similarity transformations. Then the sentence

$$(\exists a)(m_1(a)=1) \vee \neg\ (\exists a)(m_1(a)=1)$$

would have the value T although its single component atomic formula has the value μ. On the other hand, the sentence

$$(\exists a)(m_1(a)=1) \vee (\exists a)(m_1(a)=2)$$

would have the value μ as does its component atomic formulas, and thus the logic would not be truth-functional.

When we turn our attention to what happens to the set of meaningful formulas under inferences involving quantifiers the results are not as satisfactory as the sentential ones of Theorem 1. In formulating the theorem it is useful to introduce the notion of a *non-trivial* n-tuple $\mathfrak{G}$ of groups for a model $\mathfrak{M}$. By this I mean at least one G_i is such that every model $\mathfrak{G}$-related to $\mathfrak{M}$ may not be obtained by the identity transformation on the positive integers (a weaker notion of non-triviality will suffice to prove the theorem, but is not important for the purposes at hand).

Theorem 2. *Let $\mathfrak{M}$ be a model of $\mathscr{L}_n$ and let $\mathfrak{G}$ be non-trivial for $\mathfrak{M}$. Then the set of $\mathfrak{G}$-meaningful formulas of $\mathscr{L}_n$ whose closures are true in $\mathfrak{M}$ is not a deductive system (i.e., the set is not closed under logical consequence).*

Proof. (I sketch the proof without too great concern for use-mention distinctions.) Let G_i be non-trivial for $\mathfrak{M}=\langle \mathfrak{S}, A, m_1, \ldots, m_n\rangle$. Then there is an atomic formula of $\mathscr{L}_n$, a positive integer k, a model $\mathfrak{M}'$ $\mathfrak{G}$-related to $\mathfrak{M}$ and an a in A such that

$$m_i(a) = k$$
$$m_i'(a) \neq k.$$

Now the integer k may be described in $\mathscr{L}_n$ by an expression of the form "$1+\ \ldots\ +1$", and thus there is a $\mathfrak{G}$-meaningful sentence of $\mathscr{L}_n$ which is true in $\mathfrak{M}$ and which asserts that for every number x if $x=k$ then $x<k+1$. For the variable in the sentence of $\mathscr{L}_n$ corresponding to "x" we now specify a term of the form "$m_i(a)$", and we obtain a formula which is not $\mathfrak{G}$-meaningful and thus not in the original set.

The difficulties posed by Theorem 2 are real. It certainly does not seem satisfactory to have a notion of meaningfulness that is not deductively closed. In Suppes [59] I proposed taking the bull by the horns and redefining the concept of logical consequence. We may say that a formula is a *$\mathfrak{G}$-meaningful logical consequence* of a set A of formulas of $\mathscr{L}_n$ if and only if S is a logical consequence of A and S is $\mathfrak{G}$-meaningful whenever every formula in A is also.

Two problems beset this new notion of meaningful logical consequence. First, it will not in general be possible to give finitary rules of inference, because for many n-tuples of groups the notion of meaningfulness will not be decidable. Secondly, there is a problem of completeness. Putting aside the finitary problem we may ask this question. If a standard set of rules of inference is used, augmented by the requirement that each step of an inference be $\mathfrak{G}$-meaningful, does being a meaningful logical consequence imply derivability by the rules of inference? This problem has not been solved for any significant n-tuples of groups.

The discussion of empirical meaningfulness and the related three-value logic has been restricted to the simple languages $\mathscr{L}_n$, but if we are willing to move from talk about formulas and sentences to talk about propositions or relations, it is easy to generalize the results to richer physical theories. The three-valued logic of T, F and μ applies in a wholly natural way to the system-of-classical-particle-mechanics functions introduced in McKinsey and Suppes [55], for example. In this case the groups of transformations are no longer restricted to one-to-one functions from the real numbers to the real numbers, but include the geometrical and Galilean transformations familiar in classical mechanics. Other examples from classical and relativistic physics are easily constructed.

In the past thirty or forty years a large number of different multi-valued logics have been proposed, and the technical literature on the formal structure of these logics has reached sizable proportions. The philosophical defense of the significance of multi-valued logics has been much less substantial. The interest of the discussion in this paper is

meant to be in the intuitively direct philosophical defense possible for the three-valued logic introduced. The extension of classical two-valued logic developed here is too weak to raise any interesting technical problems at the level of sentential logic, and perhaps too complicated at the level of predicate logic, for which the problem of completeness in terms of meaningful logical consequence is central. But the empirical or methodological defense of the logic is clear. The data of experiments, or the empirical predictions of physical theories, are best expressed not by a particular model of the theory but by an appropriate equivalence class of models. Sentences or propositions of the theory which do not have an invariant truth value (in the two-valued sense) over these equivalence classes simply do not have any clear empirical meaning. A three-valued logic, with the third value being meaninglessness, provides a simple device for segregating these non-invariant propositions that do not have a definite empirical content, and for calling attention to the otiose features of any one model of a theory.

3. Families of Boolean algebras in quantum mechanics [1]. Turning now to the logic of quantum mechanics, I try to develop an approach that hews somewhat closer to classical lines than many discussions. I also frankly admit that many of the proposals made in the literature are hard for me to fathom from an empirical or experimental standpoint. It is a simple enough matter formally to define orthocomplemented modular lattices and relate them to the structure of subspaces that arise in quantum mechanics, but exactly how this logic corresponds to the set of experimental propositions or events is ordinarily not elaborated in any detail. For example, the much-cited article of Birkhoff and von Neumann [36] restricts the consideration of the decisive supporting experimental consequences of quantum-mechanical theory to a few lines (p. 831).

My objective here is to give a simpler and more classical logic for the experimental propositions or events. I mean by an experimental event an event that may at least in principle be observed by some configuration of experimental apparatus. In this sense the Heisenberg uncertainty principle is not an experimental proposition expressing the observation of a single event, for it is in general not possible to measure simultaneously position and momentum. In probabilistic terms,

[1] I am indebted to Dana Scott for several helpful comments on this section.

as I have emphasized elsewhere (Suppes [61a]), there is no joint probability distribution of position and momentum. It is not a matter of measuring position and probability with zero variance. It is rather that there is no possibility of measuring them jointly at all, because their joint distribution does not exist.

To simplify developments and thereby to avoid a number of technical issues, I shall make the restrictive assumption that all observables may be expressed as functions of position and momentum. Thus for a system of n particles the probability space is the $6n$-dimensional Euclidean space $\mathscr{E}_{6n}$, and I shall represent any point by $6n$ real numbers $(q_1, \ldots, q_{3n}, p_{3n+1}, \ldots, p_{6n})$, where it is understood that the first $3n$ coordinates are the position measurements and the last $3n$ the momentum measurements, with q_1, q_2, q_3 the position coordinates of the first particle, p_{3n+1}, p_{3n+2}, p_{3n+3} its momentum coordinates, etc. A one-dimensional event is just a measurable subset of points of $\mathscr{E}_{6n}$ restricted in some one dimension, and in general, an event is a measurable cylinder set $C_S \subseteq \mathscr{E}_{6n}$, where S is the set of dimensions of $\mathscr{E}_{6n}$ on which C_S is restricted, i.e., $S \subseteq \{1, \ldots, 6n\}$. The following theorem about cylinder sets is easily proved.

Theorem 3. *For any non-empty set $R \subseteq \{1, \ldots, 6n\}$, the collection of measurable cylinder sets C_S such that $S \subseteq R$ forms a Boolean algebra, which we designate $\mathfrak{B}_R$.*

The following theorem can be proved on the basis of the standard formalism of classical quantum mechanics.

Theorem 4. *Let R be a non-empty subset of $\{1, \ldots, 6n\}$. Then there exists a quantum-mechanical probability distribution on $\mathfrak{B}_R$ if and only if there is no number m such that m and $3n+m$ are both in R.*

In other words, a distribution exists on $\mathfrak{B}_R$ whenever no two coordinates representing conjugate observables are in R. It is obvious, of course, that all kinds of proper distributions exist on any $\mathfrak{B}_R$ apart from quantum mechanics. The restriction of the theorem applies only to those distributions derived from quantum-mechanical considerations. I say that a subset R of $\{1, \ldots, 6n\}$ is *admissible* whenever it satisfies the restriction stated in Theorem 4. Analogous to the definitions of Section 2, I then define a measurable cylinder set C_R as (*experimentally*) *meaningful* if and only if R is admissible. A collection $\{C_S\}$ of measurable cylinder sets is *simultaneously meaningful* if and only if there is a subset R of

$\{1, \ldots, 6n\}$ such that for each cylinder set C_S in the collection $S \subseteq R$ and R is admissible.

The departures from classical logic that follow from these definitions are evident. For example, the intersection or union of two meaningful events is not necessarily meaningful. The truth tables given in Theorem 1 certainly do not hold for this logic of meaningfulness.

My own view is that the most natural way to look at this situation is in terms of a family of logics defined in the following manner. An admissible set R of coordinates is *admissibly complete* iff it is a proper subset of no admissible set. A Boolean algebra $\mathfrak{B}_R$ of measurable cylinder sets of $\mathscr{E}_{6n}$ is a *maximal, meaningful* Boolean algebra iff R is admissibly complete. From Theorem 4 it is evident there is no unique maximal Boolean algebra. Moreover, the meaningful Boolean algebras do not form a lattice, because two maximal such algebras, of which there are 2^{3n} in the present setup, do not have a meaningful greatest lower bound or meet. On the other hand, it is evident that any two maximal Boolean algebras as defined here are isomorphic, although it would be most surprising if for any Hamiltonian the quantum-mechanical probability distributions on the two algebras were also isomorphic.

I thus come to the following thesis.

The logic of the experimental propositions of the quantum mechanics of n particles for which all observables are defined in terms of position and momentum, is the family $\mathfrak{F}(\mathscr{E}_{6n})$ *of* 2^{3n} *maximal meaningful Boolean algebras* $\mathfrak{B}_R$ *and their subalgebras* $\mathfrak{B}_S$, *with* $S \subseteq R$.

At least one of these maximal algebras is always appropriate for the logic of a given experiment, and the intersection of all those that are is the unique subalgebra of the experiment.

It may be useful to compare the logic defined here with other proposals that have been made. I first mention the three-valued logic of Reichenbach [44]. His table for conjunction is the following

$\wedge$	T	F	I
T	T	F	I
F	F	F	F
I	I	F	I

which is Kleene's [52a, p. 334] *strong* table for conjunction. But Reichenbach's three-valued logic is highly misleading, for the structure of experimental propositions or events proposed by him is not constructed

so as to be truth-functional in this logic, and the same is true of the family $\mathfrak{F}(\mathscr{E}_{6n})$ of Boolean algebras with $\wedge$ interpreted as intersection. The actual structure Reichenbach seems to be after in terms of what can and cannot be measured is rather close to what has been suggested in this paper.

By adopting a more complicated set of values, a truth-functional scheme for $\mathfrak{F}(\mathscr{E}_{6n})$ rather similar to the three-valued logic proposed in the first part of this paper may be set up. The idea is that the truth values are now not simply T, F and μ, but are ordered pairs (T, R), (F, R) and (μ, R), where R is the set of coordinates of the cylinder set corresponding to the proposition in question. To get truth-functionality, we must, as in the previous case, pay particular attention to the atomic formulas. What I mean is easily illustrated by an example. Let

$$A = \{x : x \in \mathscr{E}_{6n} \ \& \ x_1 \in (0, 1) \ \& \ x_{3n+1} \in (0, 1)\},$$
$$B = \{x : x \in \mathscr{E}_{6n} \ \& \ x_1 \notin (0, 1) \ \& \ x_{3n+1} \notin (0, 1)\},$$

where x_i is the ith coordinate of point x in $\mathscr{E}_{6n}$. Then neither A nor B is meaningful, but $A \cup B$ is, being just $\mathscr{E}_{6n}$ itself. The disjunction corresponding to $A \cup B$ is declared meaningless, for the same reasons that a similar decision was made about '$x = 1 \vee x \neq 1$' in the earlier discussion of invariance. Holding these remarks in mind, the intuitive basis for the following truth tables for negation and conjunction is evident. (I shall not formalize an elementary language to make their application precise, for it is clear from the earlier discussion how this may be done.) Because meaningfulness is preserved under complementation of cylinder sets, negation is simple.

P	$\neg P$
(T, R)	(F, R)
(F, R)	(T, R)
(μ, R)	(μ, R)

In the case of conjunction, two natural cases arise according to whether or not the set of coordinates $R \cup S$ is admissible.

Case 1. $R \cup S$ admissible.

$\wedge$	(T, S)	(F, S)
(T, R)	$(T, R \cup S)$	$(F, R \cup S)$
(F, R)	$(F, R \cup S)$	$(F, R \cup S)$

Case 2. $R \cup S$ not admissible.

$\wedge$	(T, S) (F, S) (μ, S)
(T, R)	
(F, R)	$(\mu, R \cup S)$ for all entries
(μ, R)	

A theorem like Theorem 1 may be proved for these tables. We obtain truth-functionality by widening the usual net of considerations that count in determining the truth value of a molecular sentence to include, not the full factual content of the proposition, but the set of coordinates relevant to the factual content of its atomic constituents. This inclusion marks a wider departure from classical logic than is true of the tables of Theorem 1.

The central point of comparison with the logic of quantum mechanics proposed by Birkhoff and von Neumann [36] has already been mentioned. They require that the structure of experimental propositions be a lattice, and thus that the *conjunction* of two meaningful propositions in the lattice also be in the lattice, but for reasons already stated this requirement seems too strict when one proposition expresses a possible result of measuring position and the other of measuring momentum. The fact that the closed linear subspaces of a complex separable Hilbert space form such a lattice is not sufficient, in my view, to maintain that it expresses the more restricted logic of experimental propositions, in spite of the central importance of this lattice of subspaces in the formulation of the theory.

A generalization of the logic of von Neumann and Birkhoff has been given by Varadarajan [62] and also by Mackey [63]. They define a *logic* as an orthocomplemented partially ordered set, which concept is easily made explicit. Let A be a non-empty set, let $\leqslant$ be a binary relation on A, let $'$ be a unary operation on A, and let 0 and 1 be elements of A. Let us also define another binary relation and a binary operation on A. Elements a and b are *disjoint* (in symbols: $a \perp b$) if and only if $a \leqslant b'$. And, if $a \perp b$ we define the sum $a + b$ in the expected fashion: $a + b = c$ if and only if $a \leqslant c$, $b \leqslant c$ and for every d in A, if $a \leqslant d$ and $b \leqslant d$ then $c \leqslant d$. In terms of these notions we then have:

Definition 4. *A structure* $\mathfrak{A} = \langle A, \leqslant, ', 0, 1\rangle$ *is an orthocomplemented partial ordering if and only if the following axioms are satisfied for* a, b *and* c *in* A:

1. $a \leqslant a$.
2. *If* $a \leqslant b$ *and* $b \leqslant a$ *then* $a = b$.
3. *If* $a \leqslant b$ *and* $b \leqslant c$ *then* $a \leqslant c$.
4. $0 \leqslant a$.
5. $a \leqslant 1$.

6. *If* $a \leqslant b$ *then* $b' \leqslant a'$.
7. $(a')' = a$.
8. $a \dotplus a' = 1$.
9. *If* $a \leqslant b$ *then there is a* c *in* A *such that* $c \perp a$ *and* $a \dotplus c = b$.
10. *For any sequence of pairwise disjoint elements* $a_1, \ldots, a_n, \ldots$ *of* A *there is a* c *in* A *such that for all* n, $a_n \leqslant c$ *and for every* d *in* A, *if for every* n, $a_n \leqslant d$, *then* $c \leqslant d$.

The purpose of Axiom 10 from the standpoint of probability distributions on logics is obvious; it guarantees the necessary underlying structure for a countably additive measure.

Although an orthocomplemented partial ordering is not a lattice, the requirements are still too strict for the collection of experimentally meaningful events in the family $\mathfrak{F}(\mathscr{E}_{6n})$ of Boolean algebras. The intended interpretation is that A be $\bigcup \mathfrak{F}(\mathscr{E}_{6n})$, $\leqslant$ be set inclusion $\subseteq$, $'$ be set complementation, 0 be the empty set and 1 be $\mathscr{E}_{6n}$. But this structure does not satisfy Axiom 10, because the union of two disjoint meaningful events is not itself meaningful. For example, let

$$C = \{x : x \in \mathscr{E}_{6n} \;\&\; x_1 \in (0, 1) \;\&\; x_2 \in (0, 1)\},$$
$$D = \{x : x \in \mathscr{E}_{6n} \;\&\; x_1 \in (1, 2) \;\&\; x_{3n+2} \in (0, 1)\},$$

where x_i is the ith coordinate of point x in $\mathscr{E}_{6n}$. Clearly both C and D are meaningful, disjoint events, but $C \cup D$ is not meaningful, and is thus not in $\bigcup \mathfrak{F}(\mathscr{E}_{6n})$.

The results stated for $\mathfrak{F}(\mathscr{E}_{6n})$ thus far have been rather negative, but there is a positive result related to the work of Kochen–Specker [*], namely, $\mathfrak{F}(\mathscr{E}_{6n})$ with the appropriate structure is a partial Boolean algebra. Roughly speaking, every partial Boolean algebra may be represented by a collection of Boolean algebras satisfying certain consistency conditions. For details see the article of Kochen–Specker [*].

In this sense the family of Boolean algebras constructed here originally from probabilistic considerations provides yet another intuitive argument for Kochen and Specker's concept of a partial Boolean algebra. However, the issue is not entirely simple. From a probability standpoint it is also convenient to have well defined the probability of the union of any two disjoint events, as in the case of $C \cup D$ above. No conflict with Theorem 4 exists, for the theorem is concerned with the existence of a measure on the entire algebra $\mathfrak{B}_R$, not with the much narrower question of

whether or not the event $C \cup D$ has a probability. In any given experiment, C, C', D or D' may be observed, but, of course, never the complex event $C \cup D$. For computational purposes we may then in a natural way extend the probability measure to these events.

It is not possible here to study in any detail the relations between the orthocomplemented partial orderings and partial Boolean algebras, particularly the analysis of the similar but differing notions of simultaneously measurability, one of which is primitive in the theory of partial Boolean algebras, and the other of which is definable in a simple way in the theory of orthocomplemented partial orderings.

Either one of these theories certainly provides a more natural logic than that of classical Boolean algebra by itself as a structural foundation on which to erect the probabilistic superstructure of quantum mechanics, because of the difficulties about improper or pseudo-probability distributions that arise in all too direct a fashion in the classical formulation.