PATRICK SUPPES
Director,
Institute for Mathematical Studies
in the Social Sciences
Stanford University

Learning the New Mathematics[1]

THE EXPERIMENTAL PROGRAM at the Institute for Mathematical Studies in the Social Sciences at Stanford University is providing considerable evidence that children can master a modern and enriched elementary-school mathematics curriculum. Most of the relevant testing has been done in connection with material from *Sets and Numbers*, a new series of textbooks and teacher's manuals written by the author.[2]

THE SETS AND NUMBERS MATHEMATICS PROGRAM

The central concept in the materials developed for primary grades is that of a set The concept of a number is introduced as a property of a set; for example, the number one is a property of all and only those sets with exactly one member.

Throughout the beginning materials, set operations are presented as concrete analogues to numerical operations. Union of sets and addition of numbers are presented in sequence, difference of sets and subtraction of numbers are presented in sequence, subsets and inequalities are presented in sequence, et cetera. The operations and relations of arithmetic are based upon and developed from the foundational concepts of operations on sets and relations between sets Particular attention is given to mathematical sentences, equations and the translation of English sentences expressing quantitative relations into mathematical equations. Introduction

[1] The experimental and pedagogical work reported has been supported in various phases by the Carnegie Corporation of New York, the Course Content Improvement Section, National Science Foundation, and the U.S Office of Education I am much indebted to my research assistant, Miss Diana Axelsen, for help in preparing this article.

[2] Patrick Suppes, *Sets and Numbers*, Books K-6 (Syracuse, New York L W Singer Co , 1965)

of commutative, associative and distributive laws is an important part of the third-grade content. Precise formulation of concepts and the importance of structure and laws underlying problems are emphasized as well as the acquisition of arithmetical skills.

Classroom experimentation in the *Sets and Numbers* project is now in its fifth year. Prior to this a pilot study in 1959-60 involving four first-grade classes was instrumental in the development of the first curriculum materials. During the academic year 1960-61, 25 first-grade classes in the San Francisco Bay Area were included in the program. Later the first-grade program was expanded to include 80 classes.

In 1962-63 the program included children in the first, second and third grades. The program involved 50 schools in nine school districts in the United States, with approximately 9,000 children participating. Work with the classes is being continued through the current academic year. In addition, a longitudinal study of 40 gifted students in an accelerated program was begun in 1963-64.

EVIDENCE OF LEARNING

The program of testing in the *Sets and Numbers* project has consisted of a variety of types of tests. First, achievement tests are given all experimental classes at the completion of each book in the series. These tests assess the ability of the children to learn content covered in the materials and also provide evidence of specific difficulties in learning particular concepts and skills.

In general, the achievement level on these tests is quite high. Table 1 gives the mean of scores on tests covering Books 1A through 3B, for the first three years of the classroom program. Specific analysis indicates that

Achievement Tests on Books 1A through 3B, 1960-61, 1961-62, 1962-63

Year	Test	N	Total Possible	Mean
1960-61	1A	620	73	61.40
1961-62	1A	1803	80	68.44
1961-62	2A	893	140	120.90
1962-63	1A	1236	80	72.10
1962-63	1B	172	58	49.27
1962-63	2A	1161	140	120.97
1962-63	2B	292	127	85.82
1962-63	3A	609	190	140.51
1962-63	3B	150	115	76.54

the notions of set and set operations are comprehensible to first-graders and are somewhat less difficult than operations on numbers.

Results show that the most difficult problems were two-step ones such as 2 + ... = 3 + 1, and problems which mix two types of notation. Use of variables did not increase the difficulty of work with equations. The least difficult items were invariably those using the empty set or the number 0.

ENRICHED CONCEPTS PLUS COMPUTATIONAL SKILLS

An important goal of the project is to assure that introduction of many new concepts does not decrease proficiency in work with concepts ordinarily taught in more conventional programs. Indeed, the expectation is that experimental classes will reach higher levels of arithmetical achievement than classes using traditional methods. Some test results which bear out these predictions are described below.

At the end of the 1960-61 school year, a test of general achievement was given to all experimental classes and to an equal number of comparable non-experimental classes within the same districts. The classes (which will be referred to here as control classes) were matched with the experimental classes by the districts' administrators. The administrators' choices were based upon their judgment of student achievement levels and ranges of ability, staff capabilities, comparable socio-economic background and any other relevant factors of which they were aware.

The test instrument covered the concepts found within most first-grade arithmetic books, with no material which is unique within the *Sets and Numbers* program. It included some problems based on concepts covered in any first-grade curriculum but in forms which were unfamiliar to both experimental and control groups. Thus some items required the children to apply what they had presumably learned to problem forms which were new to them

A full report of results is not included here, but a comparison of mean scores and medians is reported in Table 2 below. The total possible score on the test was 118.

Comparative Scores on General Achievement Test, May 1961

	N Students	N Classes	Mean	Median
Experimental Group	595	25	97 39	104
Control Group.....	539	25	86.68	90

The difference between means, in favor of the experimental group, was significant, with $P < 01$

In terms of the proportion of children scoring correctly on individual

items, the results can be broken down in the following ways:

(i) There were not significant differences between experimental and control groups on items involving
   a. simple recognition of groups and of Arabic numerals
   b. sequence of numerals
   c. telling time, ordinals, and fractional part (half).

(ii) The experimental group was superior on items involving
   a. decomposition of tens and ones, place value
   b. writing numerals

   and markedly superior on
   c. items involving arithmetical operations (addition and subtraction, in column or in equation format, in several degrees of complexity and in several forms).

Standardized test results[3] obtained in 1961-62 indicate that children using the *Sets and Numbers* materials scored at least as well or better on the traditional content as children who had been in conventional programs. In addition, a considerable body of content not tested by standardized tests was taught.

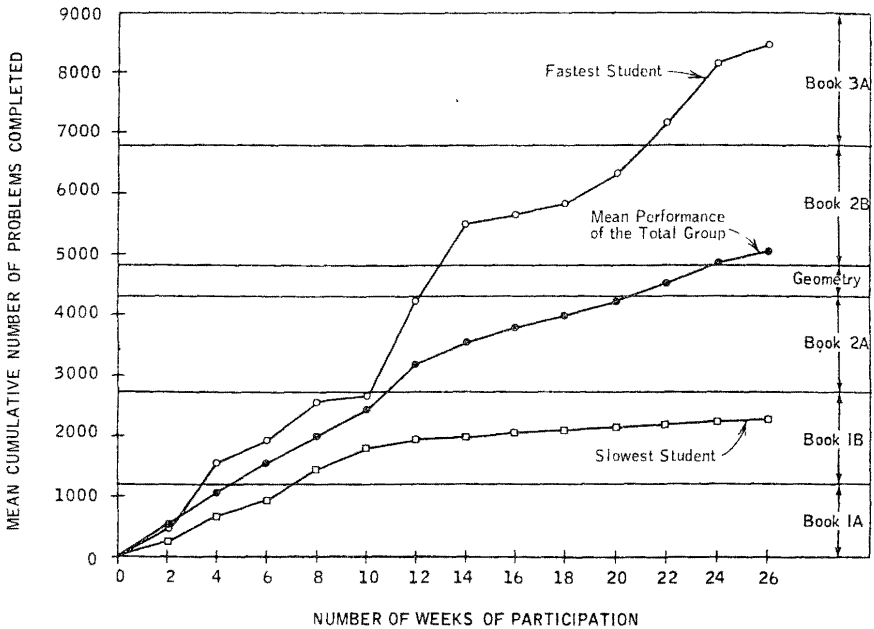
LONGITUDINAL STUDY OF GIFTED CHILDREN

Further evidence of a child's ability to handle a new mathematics curriculum has been provided by a longitudinal study of gifted children in an accelerated program. The project, begun in 1963-64 with a group of 40 first-graders, will cover the first six years of the students' mathematics training.

The initial group was selected by giving the *New York Test of Arithmetical Meanings* to all the children entering the first grade in four schools. Those above the seventieth percentile on this test were then given general intelligence tests.

The marked differences in learning rates for these children is one of the most important results of the study. All the students are proceeding through the curriculum at a rate far above the initial expectations of the project staff. For example, the group mean of problems completed daily was 154.61 for the initial 26 weeks of instruction, and the mean performance of the group represented the completion of approximately one and three-quarters years of the curriculum. Figure 1 depicts the over-all performance in terms of a group cumulative curve. A calibration of the number of problems in terms of the *Sets and Numbers* text materials is shown on the ordinate.

[3]Metropolitan Achievement test, Arithmetic Section, Primary I and Primary II Battery.



The results of three achievement tests administered during 1963-64 to the accelerated students also indicate a very high level of performance. The appropriate *Sets and Numbers* achievement test was administered two weeks after a child finished the particular text material. The SRA tests, Achievement Series, level 1-2, and Greater Cleveland Achievement Tests, Grade 1, parts I-IV, were given in the last week of May 1964.

The SRA tests included items that were novel in format and had unfamiliar notation. Although the children were given no special instruction or training for the test, they still performed in the top quartile in comparison to normative group results that were reported in the test manual.

THE PROGRAM IN MATHEMATICAL LOGIC

An experimental program in the teaching of mathematical logic affords further evidence that elementary school children can master concepts of modern mathematics not previously included in the curriculum. Since 1960 a project has been conducted under the author's directions with fifth- and sixth-graders in San Francisco Bay Area schools. The materials developed for the classes have now been published.[4]

The students are introduced to the theory of proof and to the methodology of deriving theorems from axioms. In addition to learning the

[4] Patrick Suppes and Shirley Hill, *First Course in Mathematical Logic* (New York Blaisdell Publishing Co., 1964).

formal theory of logical validity and consistency, the student also learns to recognize the logical structure of English sentences.

Achievement tests have been given to these elementary school students and to two Stanford University logic classes which served as control groups. The experimental and control groups were composed of students approximately equal in native ability.

The examinations suggest that the upper quartile of elementary school students can achieve a significant conceptual and technical mastery of elementary mathematical logic. The level of achievement is 85 to 90 per cent of that of comparable university students. Elementary school students need the same amount of work time but allocated over a longer period and with considerably more direct teacher supervision.

ADDITIONAL RESEARCH RESULTS

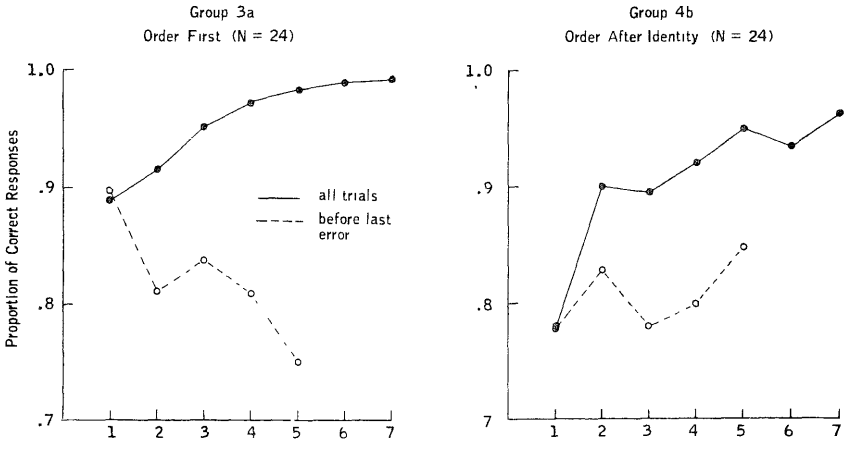
The research program directed by the author also includes a wide variety of experiments in mathematical concept formation The results of some of these studies are of interest in the present context, since they deal with the learning of mathematical concepts by young children. There are experiments concerning geometrical properties, set-theoretical concepts, binary numbers, and proofs in a simple mathematical system, where logic is treated in some cases as a formal uninterpreted system and in others as an interpreted one.[5] A brief report is included here on an experiment which required first-grade subjects to learn the concepts of set equivalence (the concept of sets having the same members), identity of sets and identity of ordered sets.

Ninety-six first-grade children were used, in four groups of 24 each. In Group 1 the children were required to learn identity of sets for 56 trials and then equivalence for a further 56 trials. In Group 2 this order of presentation was reversed. In Group 3 the children learned first identity of ordered sets and then identity of sets. In Group 4 the order was reversed. No stimulus display on any trial was repeated for an individual subject; this procedure guaranteed that the learning could not be explained by simple stimulus-response association.

The mean learning curves for Group 3 on their first task and Group 4 on their second are shown in Figure 2.

[5] Patrick Suppes, "The Development of Mathematical Concepts in Children," Technical Report No. 63, Psychology Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, February 25, 1964 a, 50 pp

Patrick Suppes, "Mathematical Concept Formation in Children," Technical Report No. 64, Psychology Series, Institute for Mathematical Studies in the Social Sciences, Stanford University, April 10, 1964, 30 pp



As is evident from these curves, the number of errors on the concept of identity of ordered sets was extremely small. From the high proportion of correct responses even in the first block of trials of this concept, it is obvious that this is a very natural and simple concept for children.

SOME TENTATIVE PEDAGOGICAL CONCLUSIONS

Research on how children learn the new mathematics and accumulation of evidence as to the range of concepts it is feasible for them to acquire has only begun. It is certainly too early to attempt any definitive conclusions about what is possible or practical. However, it *is* reasonable to draw certain tentative conclusions from our research over the past seven or eight years, and I should like to present such a summary here.

(1) The ability to learn a wide range of modern mathematical concepts is not restricted to gifted students. The *Sets and Numbers* program has been used with a wide variety of classes in a wide variety of schools. Moreover, for three years the program had been used with several classes in Ghana, and children with quite restricted backgrounds have made very satisfactory progress.

(2) By placing an emphasis on understanding and a simple presentation of concepts, the new mathematical programs have been able to

achieve a high degree of rigor and yet remain pedagogically simple. It is to be understood that this simplification is itself a by-product of the much greater clarity about the nature of mathematics and the fundamental concepts of mathematics which has been achieved in the twentieth century. Readers who are skeptical of this point are advised to read any book on the teaching of arithmetic written in the nineteenth century. In many cases the practical advice in these books is reasonably good. The mathematical discussion of the foundations of arithmetic is almost uniformly bad. A good example is the widely known book by McLellan and Dewey, *The Psychology of Number and Its Applications to Methods of Teaching Arithmetic,* which was written in 1895. The psychological parts written by Dewey are quite reasonable, but from a modern standpoint the discussions of arithmetic are most unsatisfactory.

(3) By placing an emphasis on understanding, children may often learn the new mathematics more easily and more rapidly than was observed in the past when emphasis was often placed on rote skills.

(4) One of the most difficult pedagogical problems for teachers is how to arrange their teaching to accommodate the great variation in ability among their students. It is our own judgment that this is perhaps the most important problem now confronting elementary-school mathematics programs. In one sense it is interesting to find that a problem of this fundamental importance is a teaching problem and not a mathematics or curriculum problem. The practical advice to teachers, at least in our own experience, is to move toward a greater degree of individualization of instruction whenever possible. Many of the new mathematics programs are written in a more self-contained fashion than the older programs and thus permit a greater degree of individualization.

(5) The mathematics curriculum will continue to change and deepen for the indefinite future under the impact of the many pedagogical and psychological experiments now being conducted in all parts of the world. It is a mistake to think of any current curriculum as being the final word on how elementary-school mathematics should be organized and taught.