Experimental teaching of mathematical logic in the elementary school*

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The problem

For many years, especially since the successful launching of Sputnik I, there has been a great concern to upgrade the mastery of mathematics by students before they reach the college level. Initial efforts were directed toward a revision of the high school mathematics curriculum.

One primary goal was to introduce the high school senior to the concepts and skills of the calculus and analytic geometry, and to prepare him to use the calculus in college freshman physics. This aim was suggested by the Commission on Mathematics of the College Entrance Examination Board in its 1958 *Mathematics for Today*. The Commission also recommended that the high school student be introduced to basic concepts of mathematics, such as the different kinds of numbers, inequalities, sets, variables, functions, and relations, and to the nature and role of deductive reasoning in algebra as well as in geometry.

In recent years no one has supposed that high school students are incapable of grasping such concepts. According to early views of the problem, the task was simply to raise the level of student achievement during the high school years from the average achievement upon entrance

Attempts to carry out this program involved, first, allotting more time for mathematics in the high school curriculum and, second, increasing the efficiency of mathematics instruction. The mathematics curriculum was improved by eliminating nonessential topics and by emphasizing concepts basic to the various branches of mathematics. With the introduction of these more fundamental notions, the traditional picture of mathematics as a collection of separate topics gave way to a treatment of mathematics as the study of sets of assumptions and their deductive consequences. Hopefully, this more unified approach will make each stage of study more successful, and also will prepare the student more adequately for future work. So far, it appears that the changes are helping to bring about these objectives.

In recent years a third approach to the problem has developed: Attempts are being made to raise the level of mathematical achievement before the student

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enters high school. This is done by introducing topics previously not taught before high school and by emphasizing the logical nature of mathematics.

The central task of the present project was to explore the possibility of teaching deductive techniques to gifted elementary school children. In particular the project investigated the difficulty and suitability of teaching the elements of mathematical logic to fifth- and sixth-grade students.

Objectives and hypotheses

The encouragement of critical thinking is generally acknowledged as one of the legitimate objectives of our schools Yet there has been little effort to translate this goal into behavioral terms. If this objective is to be realized, a clear notion of the relevant skills must guide the selection of classroom subject matter and teaching methods. Critical thought requires the ability to make logically correct inferences, to recognize fallacies, and to identify inconsistencies among statements. If the school is to encourage these skills significantly, instruction must deal specifically with the ability to derive logical conclusions from given sets of premises, evidence, or data.

Through the study of logic, the child can also be introduced to clear and precise language usage. Emphasis can be given to the dual goals of clear, rigorous thought and of precise and effective formulation of ideas. The student can be introduced to the concept of logical structure in language; thus, he learns to recognize and analyze the forms of sentences. He also can come to understand the connection between formal structure and logical validity. Although the present project was considered a supplement to the elementary school mathematics program, the application of the concepts and skills learned is not limited to this subject. The principles of logical inference are applied in every branch of systematic knowledge. A class in formal logic is an opportunity to pay more than incidental attention to the development of the student's reasoning powers.

Advances by modern logicians and mathematicians have led to a systematic treatment of the relations between logic and mathematics. A completely explicit theory of inference has been formulated to deal with all the standard examples of deductive reasoning in mathematics and the empirical sciences.

The teaching of mathematical logic can not only develop general reasoning ability, but also provide for a deeper and more penetrating study of mathematics and mathematical methods. The student can learn to approach mathematics as a study of axiom systems and the derivation of theorems from axiom systems. This project assumed that the study of deductive methodology and the theory of proof can enrich the mathematical experience of the elementary school child. The subject matter at this fundamental level of mathematics was introduced in a rigorous fashion; yet the presentation and context were simple enough to permit comprehension by at least the more able elementary school students.

Specifically, the objective of the project was to experiment with teaching mathematical logic to classes of academically talented fifth- and sixth-grade students Thus, it was a program for the enrichment of the more gifted students. A question of particular interest involved the capacity of children at the fifth- and sixth-grade age levels to do deductive proofs. Other questions concerned specific factors of difficulty, and the possible transfer of skills in analysis and correct reasoning to other subject areas, such as arithmetic, reading, and English.

The implementation of these objectives immediately raised the problem of teacher preparation. Thus, a secondary goal was the organization of a teacher-training program. It was necessary to determine the amount of preparation a teacher needs to present adequate instruction in logic to fifth- and sixth-grade children.

Related research

Research on the capacities of children of particular ages to reason logically is scarcely definitive. Inhelder and Piaget [3][1] have concluded that development of capacity for "hypothetical" reasoning or formal aspects of logic begins at about age eleven. However, a recent study by Hill [1] indicates that children of ages six, seven, and eight have a considerable intuitive grasp of many principles of logical inference, and, further, they can demonstrate their understanding in reasoning from hypothetical premises. Her results also indicate that simple demonstrations of correct deductions improve children's performance in the recognition of valid inference. A study by Hyram [2] also has suggested that improvement in children's inferences may be brought about by direct instruction in the logical rules.

Procedure

Pilot study, 1960–62

In the autumn of 1960, a pilot study in the teaching of mathematical logic was begun at an elementary school near Stanford University. The experimental group consisted of 25 students selected from all the fifth-grade classes on the basis of arithmetic achievement scores. The children had been given the arithmetic test during the seventh month of their fourth-grade year. Only students who had scored at or above the sixth-grade level at that time were chosen for the logic program. The class comprised about 27 percent of the fifth-grade students in the school.

The class met three days per week throughout the school year for sessions of 30 to 35 minutes. The class was taught in 1960–61 by Dr. Shirley Hill and in 1961–62 by Mr. Frederick Binford, both project staff members.

The class used textbook material prepared by the project staff and composed of explanatory material and practice exercises.

Teacher preparation

During the summer of 1961 ten teachers and three school principals participated in an intensive four-week training session. The materials planned for the elementary schools were used in this course. Another fourteen teachers took a night school course in the spring of 1962.

Seventeen teachers from the first course enrolled in a second class in the fall of 1962. At the same time they were teaching their own first- or second-year classes in logic to fifth- and sixth-grade children.

Again, in the spring of 1963 a teacher preparation class was offered, with twenty-six teachers enrolled. Sixteen of the participants were given a second course in the fall of 1963.

Experimental classes, 1961–62

Twelve classes with a total of 350 fifth-grade students began their study of logic in the fall of 1961. These classes represented 11 different schools in six school districts of the San Francisco Bay area. In ten schools the school administration selected approximately 25 to 30 percent of the fifth graders for the logic program on the basis of general ability and high achievement in mathematics. In the other two schools all of the fifth graders were given logic; in both schools, however, it was necessary within a few months to divide the classes into fast and slow sections.

Most of the classes met three times a week for about 30 minutes, although a few met more often or for longer sessions. Each of the classes was taught by a classroom teacher from the staff of the school, with one exception: one class was taught by Mr. Binford of the project staff.

The work in the experimental classes largely followed the sequence presented in the textbook. No effort was made to maintain a uniform rate of progress in all groups; rather, each teacher proceeded at a pace that seemed appropriate for that

[1] Numbers in brackets refer to the References at the end of the article.

class. During the year these classes completed between 115 and 195 pages of the text, *Mathematical Logic for the Schools*, by Patrick Suppes and Shirley Hill.

Experimental classes: 1962–63

A second year of logic was completed by 11 sixth-grade classes consisting of 215 students who had their first year of logic in fifth grade (1961–62). Most of the classes were reviewing throughout the autumn. By June, they had completed between 162 and 284 pages of the text.

In addition, 12 new classes of 269 fifth-grade students began their first year of logic and completed between 117 and 183 pages of the text. One class began late in February with 18 pupils and completed 60 pages of the text.

Control groups

The control groups were two Stanford University logic classes, one of which met in the spring and the other in the autumn quarter of 1962. These students were taking logic at the customary age and educational level; the mode was between the sophomore and junior year of college. The text prepared for the elementary school students was used and the autumn-quarter class was taught by Professor Suppes.

It is fair to assume that the native ability of the elementary school children selected for the program is comparable to that of the Stanford University students.

Testing program

The achievement of each class, experimental or control, was measured by a series of tests administered to each group whenever it had reached the appropriate point in the text.

The 1961–62 series consisted of two tests, one administered when the class had completed the standard rules of derivation, the other when the class completed truth tables. These tests were scored for each class by the teacher, who followed carefully prepared instructions and consulted with the project staff.

The 1962–63 series consisted of three tests, the first two covering the same material as the tests for 1961–62 and the third covering the first two chapters of predicate logic. These tests were objectively scored.

Curriculum material

During the summer of 1961, the materials developed by Professor Suppes and Dr. Hill for the pilot class of 1960–61 were extended and revised with suggestions from Mr. Binford. This text material, titled *Mathematical Logic for the Schools*, was used for the classes in 1961–62.[2]

On the basis of the experience of teachers and children, the text was revised again in the summer of 1962. Further exercises were added, some changes in rules and form were made, and some of the materials were arranged in different order.

In the summer of 1961, Mr. Binford and Dr. Hill prepared a manual of solutions for all of the exercises in *Mathematical Logic for the Schools*. In the summer of 1962, Mr. Binford revised this manual to accompany the revised text; also, comments, suggestions, and explanations were added for the teacher.[3]

Supervision

The project staff gave the teachers regular supervision. In 1961–62, a member of the project staff visited each class about every six weeks, to observe, to confer with the teacher, and to do some of the teaching. Also, regular monthly meetings were held with the teachers.

During the autumn of 1962, the teachers were taking the second course and thus were meeting with the project staff every week. For the remainder of the year a member of the staff visited each class and conferred with the teacher about every three weeks.

[2] An outline of the text appeared in THE ARITHMETIC TEACHER, IX (November, 1962), 398–399.

[3] Still additional revisions in the text were made subsequent to the project, and the revised version was published in 1964 under the title, *First Course in Mathematical Logic*, Blaisdell Publishing Co. Mr. Binford's manual was also revised and published at the same time.

Special class, summer 1963

A special follow-up class was offered to the most successful of the elementary school children who had studied logic in both the fifth and sixth grades. Twenty-six students were selected upon the recommendation of their teachers and by a special examination given the candidates. Joining the class was entirely voluntary for the children.

Mr. Binford taught the four-week course, which met five days a week for one-hour sessions. In addition the students did outside assignments. The class used new materials which continued the text prepared for the regular course. These materials presented the logic of equivalence and developed the theory of multiplication and division from standard axioms.

At the end of four weeks a voluntary two-week extension of the special class was announced. Ten of the students continued. During these two weeks they studied the algebra of sets and some elementary topology. From Huntington's axioms for Boolean algebra they proved a number of theorems. Then, with four additional axioms on topological closure, they proved a number of elementary theorems of topology.

Analyses of test data and other findings

Our examinations of achievement in mathematical logic indicate that the level of accomplishment of elementary school students is comparable to that of a college class. However, the college students put in about ten times as many hours each week on logic as did the elementary school students. So, although the college students completed in four weeks the material that the elementary school students covered in an academic year, their achievements per hour were comparable. Comparisons were not made of the relative progress of college and elementary school students at specified times. Rather, each group was tested when it reached particular points in the text material. Thus, mastery of course content by experimental and control groups could be compared.

Two tests were given to the college group of 186 students in the spring of 1962. The first of these tests was also given to the 260 students in the fifth-grade classes. The examination dealt with the recognition of the logical structure of sentences and with early work in symbolic deduction.

The results of Test I are given in Table 1. A few very low scores pulled the mean well below the median. Both mean and median are given in the table.

Table 1
Summary test results for 1961-62

	N	*Mean*	*Median*	*Range*
		(*possible score*, 100)		
Test I				
College students	186	91	97	8–100
Fifth-grade students	260	76	83	17–100
Test II				
College students	189	70	72	19–99
Fifth-grade students	164	67*	77	6–100

* The mean scores of the seven classes taking Test II were 92, 87, 79, 78, 76, 49, and 36.

The elementary school classes progressed through the material at various rates. Seven of the classes with a total of 164 children got at least far enough to take Test II. This test was concerned with the symbolic manipulations involved in derivations. Table 1 shows that their median was above that of the college group. Two of these classes had very low mean scores, but the other five were well above the college mean. (See footnote to Table 1.)

These test results suggest that the symbolic manipulations performed in formal proofs are relatively easier for elementary school children than are the problems of recognizing the logical form or structure of sentences in ordinary English.

The testing program continued in 1962–63. Tests were administered to another control group of college students, to the experimental classes of students who had

Table 2
Means and ranges for scores for Test I, 1962-63

	By section of test				Total Test
	A, B, C	*D*	*E*	*F, G*	
Possible score	26	21	12	26	85
Mean					
Control	22.96	20.25	11.05	24.75	79.0
Experimental	19 0	17.0	10.1	22.9	69.5
Range					
Control	26–15	21–7	12–2	26–0	85–24
Experimental	25–5	21–7	12–5	26–4	84–32

Table 2A
Ratios of mean score, elementary school to college

	By section of test				Total test
	A, B, C	*D*	*E*	*F, G*	
All experimental students to control	.83	.83	.90	.94	.88
Best experimental class to control	.96	.89	1.09	1.04	.975

advanced to the sixth grade, and to beginning classes of fifth-grade students.

The 1962–63 edition of the tests was made out in multiple-choice form with answers to be transferred to a standard answer sheet. This was done to allow rapid objective scoring.

Sections A, B, and C of Test I covered the translation of English sentences into logical symbols, the construction of negations, and the classification of different logical forms of sentences. These are largely language skills. Section D dealt with application of rules of inference to one-step derivations. Section E tested logical vocabulary, and Sections F and G called for construction of multiple-step derivations.

Tables 2 and 2A summarize the findings on Test I in 1962–63. They show that the elementary school children, as compared to the college control group, did relatively better on the pure, symbolic, logical deductions (Sections F and G) than on work requiring language skills (Sections A, B, and C). Similar results were obtained in 1961–62. Except for question 41, which none of the elementary group answered correctly, the ranges for each section of Test I were nearly identical for the two groups. Perfect test results were obtained by some students in both groups, and the best experimental class was quite comparable to the college class.

Section A of Test II (see Table 3) covered deductions in multiple-step problems. Sections B, C, D, and E tested the ability to do proofs of invalidity, of consistency, and of inconsistency. Ten of the experimental classes, totaling 196 students, completed the test.

Three of these classes with a total of 60 students also completed Test III, which covered translation of English sentences into the symbolism of predicate logic and the carrying out of derivations in this symbolism.

On Test II the mean of the best class of elementary school children almost exactly matched the college mean. The ranges for experimental and control groups were identical; several perfect test papers were obtained from both groups.

Three experimental classes completed Test III. One college student handed in a perfect paper, while one elementary school

Table 3
Results of Test II and Test III, 1962-63

	Test II			*Test III*	*Total of Test II and Test III*
	A	*B–E*	*Total*		
Possible score	28	33	61	41	102
Mean					
College	27.3	25.8	53.1	27.7	80.0
Elementary school	25.2	19.7	44.2	26.0*	70.2*
Range					
College	28–12	33–6	61–18	41–13	102–31
Elementary school	28–12	33–3	61–19	39–15*	100–34*
Ratio of means:					
Experimental to control	.92	.76	.83	.94*	.87*

* These figures are based on the three classes (60 students) that took both Test II and Test III

student missed only two points. Table 3 summarizes the results on Tests II and III.

Comparisons between the 1961–62 and the 1962–63 test results require interpretation. The results from Test II in 1961–62 and Tests II and III in 1962–63 are of particular interest. In 1961–62 the elementary school median was 7 percent above the college median, while in 1962–63 it was 15 percent below. The teachers administering the latter tests reported that the students had difficulty handling the mechanics of the objective tests and were often confused by the multiple choices. One teacher whose class had a median of five points out of 15 on the last section of Test III later gave the students the same problems to work without any right and wrong suggestions. She reported that the great majority wrote perfect original solutions, many of which were highly idiosyncratic. Without doubt the college students were a great deal more practiced in the mechanics and strategy of multiple-choice examinations and more readily accepted a given, fixed approach to a proof.

Intelligence quotients

Scores on the California Test of Mental Maturity were available for 176 of the students who began logic in 1961. The lower quartile ranged from 98 to 122, the second quartile 123 to 130, the third quartile 130 to 142, and the fourth quartile 142 to 184.

Subjective evaluations by teachers

In June, 1962, the teachers in the program were asked for their subjective evaluations about what general benefits to the students could be attributed to the logic program. They were asked to be frank in giving comments of their own and comments they had heard from others.

The following replies were received.

Reading growth: the class *median* growth showed a gain of more than two grade points as shown by Iowa tests. Confidence in meeting new symbolisms or new systems of notation was realized

Enthusiasm of the students for the program and support of the parents showed that the subject matter is interesting to above-average students. It offered good experience in vocabulary building by a greater awareness of language and meanings. Students began to challenge careless statements by the teacher.

A more precise way of attacking problems was acquired by some of the students. A lively interest appeared in solving problems in an original, individual kind of way on the part of some students. The excitement of a new subject in the curriculum at this grade level stimulated the students.

I have thoroughly enjoyed teaching this first year of mathematical logic. It has been a challenge to me, and it has been a rewarding experience to see children of this age respond intelligently and enthusiastically to a whole new way of thinking.

This experience has not been, however, a carefree adventure. Aside from the fact that it was a new subject for me to present, the time-consuming job of reviewing each day's assignments was sometimes a great burden. This was particularly true when we reached long proofs where the individuality of approach to solving problems became apparent. Absences of two or more days often required coaching outside of class time. The suggestion that we use some of the abler students to assist the teacher was a good one and helped to some extent, but not greatly.

In spite of these reservations I am looking forward to going on next year.

I think it is interesting to note some of the results of this last test. Two of the outstanding students were sixth graders, but the low scorers were also sixth graders. The fifth graders generally did much better. I have tried to analyze this difference and have concluded that it may be attributed to a difference in attitude toward this subject and possibly to other subjects. I discussed some of these children with their homeroom teacher and learned that, with one exception, the low, low scorers in math logic were not performing up to their ability in other subjects. Another thing that may be considered in evaluating this class is the richness of this last year's curriculum. In the field of math alone they were exposed to new concepts not usually encountered before the eighth grade and high school—and so in every other subject. Therefore, their interest was not centered in math logic.

On the other hand, in Room 12, composed of the more able students of the two fifth grades, math logic was the new subject introduced into the regular fifth-grade curriculum. They were very proud to be studying it, and most of them worked diligently. In this last test, their median was seven points less than the other class of higher ability and their average only 2.3 points lower.

The program develops creative and critical thinking, responsibility for homework, provides challenge and stimulation to the advanced child, gives prestige in home and school to children who are not always as socially accepted as others.

There were no adverse comments from parents. They appeared to be flattered and curious as to the outcome. I think it made the pupils aware of the fact that there are other areas of learning than routine arithmetic, spelling, and reading.

Parents are impressed and interested. I enjoy teaching the program also.

Not very much transfer seemed to occur. I particularly stressed transfer when discussing *errata*, and students brought examples all year long. There was greater language awareness.

I found enhanced self-confidence in achieving a mastery of a kind of discipline that some students originally thought could not be mastered by them. They acquired a more critical attitude toward statements they ran across; a recognition of the need of analysis in complex situations; and an awareness of sentence structure in written language. Students who at the beginning seemed to have difficulties came through with good work eventually.

Signs of transfer were more apparent among the children with exceptional ability. They were very apparent in arithmetic. One child remarked that a section of his Reading Achievement Test was made easier because of the methods of logical interpretation he had learned. There was also some frustration in their trying to apply their knowledge or methods in other fields.

There has been a transfer in arithmetic. One helps the other. There is also a growing awareness in English grammar as the two are compared.

Conclusions and implications

Our summary conclusions are the following:

1 The upper quartile of elementary school students can achieve a significant conceptual and technical mastery of elementary mathematical logic. The level of mastery is 85 to 90 percent of that achieved by comparable university students.

2 This mastery of the subject matter by elementary school students can be accomplished in an amount of study time comparable to that needed by college students if study is allocated over a longer period of time if the students receive considerably more direct teacher supervision.

3 The more dedicated and able elementary school teachers can be adequately trained in five or six semester hours to teach classes in elementary mathematical logic. It is probably essential that this teacher-training program be very closely geared to the actual program of instruction the teacher will follow in the classroom.

4 Anecdotal evidence from teachers suggests that there is some carry-over in critical thinking and attitude into other fields, especially arithmetic, reading,

and English. More explicit behavioral data on carry-over in critical thinking would be desirable.

5 The work with the special summer class in 1963 indicates that able elementary school students who have received prior training in mathematical logic can make rapid progress in other parts of modern mathematics organized on a deductive basis. We hope to be able to continue this line of investigation in the future and to determine to what extent the deductive methodology and viewpoint of modern mathematics may be made a natural and intuitive part of the young student's mathematical knowledge.

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