The Stanford Institute for Mathematical Studies in the Social Sciences

APPLIED MATHEMATICS AND STATISTICS LABORATORIES
STANFORD UNIVERSITY

Reprint No. 53

Experimental Studies of Mathematical Concept Formation in Young Children

PATRICK SUPPES AND ROSE GINSBERG

Reprinted from
Science Education
Vol. 46, No. 3, April 1962

[Reprinted from SCIENCE EDUCATION, Vol. 46, No. 3, April, 1962]

EXPERIMENTAL STUDIES OF MATHEMATICAL CONCEPT FORMATION IN YOUNG CHILDREN *

PATRICK SUPPES AND ROSE GINSBERG

Institute for Mathematical Studies in The Social Sciences, Stanford University, Stanford, California

DURING the last three years we have performed a series of experiments in the area of children's mathematical concept formation, a number of which we present in this report. One of our most urgent concerns during this period and one which will probably be of close interest to our readers, has been to try to bring to this somewhat amorphous area precise quantative analysis. To achieve this we have made use of a quantitative theory of psychology, mathematical learning theory [1], which in one form or another has had considerable success in dealing with other behavioral areas [2].

To familiarize the reader with this approach we present, in the following pages, an intuitive description of the assumptions, implications and outcomes of the particular mathematical model employed and have tried to give enough of the theoretical analysis to give the reader some feeling for the possibilities of this kind of approach.

At the same time the empirical variables introduced in these experiments have not been arbitrarily chosen. This experimental project has been run concurrently with the development of a program of primary grade mathematics "Sets and Numbers" [3] which makes use of simple set theoretical notions to introduce the child at the outset to concepts and operations in arithmetic. Consequently the concepts to be learned in each experiment and the experimental variables introduced have some bearing on problems which have arisen through the development of the *Sets and Numbers* program. Because of shortness of space these experiments are presented only in summarized form and for the same reason we have limited ourselves strictly to rigorous experimental studies. Other studies involving whole classes of children, which have been run to answer some specific problem encountered in developing the *Sets and Numbers* program are not reported here.

* Based on paper presented at meeting of Section Q, American Association for the Advancement of Science, Denver, Colorado, December 29, 1961.

THE MODEL

The situation for which the model was developed is one in which a subject is presented on each trial with some stimulus display to which he is required to make a specific response. The subject is always told whether he was correct or incorrect on that trial.

The basic assumption of the model is that in such a situation the correct response will with some probability become associated with, or conditioned to, the stimulus display. If it is conditioned the subject will then, as long as the situation remains unchanged, continue to respond correctly whenever the same stimulus is presented. On the other hand where the correct response is not conditioned to a stimulus display the probability of a correct response is at some chance level which remains constant until the association between stimulus and appropriate response has been established.

To make the theoretical assumptions perfectly clear, consider a paired-associate situation. In this case the subject must learn the correct response to each of a list of stimuli. For example let us say that he must give a specific number response each time we present him with one of a list of nonsense syllables. He must respond with

"2" to the stimulus "tux," with "9" to "fax," and so on. According to our theory, if we consider the responses over trials to some particular item in the list of nonsense syllables, the probability of a correct response will be at some chance level, in other words, the subject will simply be guessing, until on a particular trial conditioning is effected, after which the subject will make no further errors.

Each time therefore, we present the stimulus "tux" the subject makes his guess and is told that the correct response is "2." On some specific trial the association between "tux" and "2" is made and after that, whenever we present "tux" the subject responds correctly. An interesting feature of these very simple assumptions is that if we look only at the responses made by each subject before his final error we know that at least up to that trial the subject has not learned the correct response but has been guessing. The implications of this particular aspect of our theory will be discussed later in this paper.

From the above assumptions, given mathematical expression, it is possible to derive a large number of specific quantitative predictions. For example, statistics for the number of errors such as number of errors before the first success, error runs of length k, number of alternations of success and error can all be estimated as can such sequential statistics as errors on trial n and on trial $n + k$.

Such theoretical predictions provide a sound quantitative basis for comparison with observations taken directly from the data. The advantage of a quantitative rather than the qualitative approach usual to this area is that on the one hand it ensures a more rigorous analysis and on the other permits detailed evaluation of the empirical adequacy of the theory of concept formation used.

THE EXPERIMENTS

Of the experiments summarized below all except the first have in common both the kind of concepts to be acquired and certain experimental methods. Much of the description therefore will be common to these experiments. Experiment I on the other hand, representing a more or less classical type of concept formation experiment is presented by itself.

EXPERIMENT I

This experiment has been presented in considerable detail elsewhere [4]. In essence equal numbers of kindergarden and first grade children were required to learn the binary equivalents of the arabic numerals 4 and 5 (100 and 101). Using different symbols for the binary digits 0 and 1 (such as π, σ, etc.) six completely different stimuli were presented one at a time in random order. The child was required to respond by placing a large 4 or 5 directly upon the stimulus. All the children were told on each trial whether they had made the correct or incorrect response and half of them were also required to correct their responses following an error.

Our main interests in this experiment were firstly, to examine the effect upon learning of requiring the subject to correct overtly a wrong response and secondly, to determine if the simple model described above which had been so successful with paired-associate learning, would account equally well for concept learning.

Our results indicated that the children who do make the immediate correction response after an error perform significantly better than those who do not. This result, in a situation where there are only two possible responses, is contrary to that obtained with adult subjects [5], but is similar to results obtained with lower organisms [6].

The fit of the model described above to the data of this experiment was good. A large number of predicted and observed statistics have been presented for these data and as an example a few of the typical results are presented in Table I.

TABLE I

PREDICTED AND OBSERVED STATISTICS FOR GROUP (N=24) MAKING OVERT CORRECTION RESPONSE. EXPERIMENT I (BINARY NUMBER)

Statistic	Predicted	Observed
1. Expected errors per item per subject	....	4.38*
2. Standard deviation of errors per item per S	3.13	3.00
3. Expected errors before first success	0.92	0.94
4. Standard deviation of errors before 1st success	1.30	1.29
5. Expected number of success runs	2.82	2.83
6. Expected number of error runs	2.44	2.48
7. Expected error runs of length 1	1.36	1.48
8. Expected errors on trial n and on trial $n+1$	1.94	1.86
9. Net difference between success following error and error following error	0.38	0.48

* Used to estimate the single parameter of this model.

In the experimental situation described, as with most concept formation situations, particularly those involving young children, each concept was represented by a small number of different stimuli. In this situation it is possible that the child simply acquires associations between specific stimuli and their appropriate responses, in which case we have essentially a paired-associate situation, which in itself might account for the good fit obtained with the theory employed. In all subsequent experiments summarized in this report therefore, the stimulus displays representing each concept are different on each trial. In such a situation with no stimulus display ever repeated, the question of whether the child can acquire the concepts involved, by recognizing a common property of a number of different displays, is of great empirical interest. Moreover if the data of such an experiment can be adequately described by the simple theory outlined above, quantitative predictions will be possible in an area which has not hitherto lent itself to precise description of this kind.

As our basic interest is in the formation of mathematical concepts, the concept of number is of fundamental importance and in all further experiments this concept was the primary learning task. Explicitly we presented to a child on each trial pictures of two sets of objects. If the number of objects in each set was the same, that is if the sets were equipollent, one of two available answers was experimentally defined as "correct"—if the sets were non-equipollent the alternative response was correct. As a further point of empirical interest, in no single set was there any duplication of objects—a set for example, might consist of a ball and a box—but never of two balls or two boxes. Following our findings in Experiment I—that learning was more rapid when the child was required to make an overt correction response after an error—we included this requirement in all the experiments reported below.

For the following experiments, no attempt will be made to give details of the experimental situation or methodology. Some of the experiments have been reported elsewhere in very considerable detail and where these are available the references will be given. In this report only a very general description of experimental situations and empirical results is offered. In a final theoretical section we will attempt to indicate a few of the interesting quantitative results which are more or less typical of the various situations examined. As indicated earlier in this paper, some interesting implications of the theory used will be discussed and evaluated in terms of the empirical evidence available.

EXPERIMENT II

The learning tasks involved in this experiment were equipollence and non-equipollence of sets and the two related concepts of identity of sets (sets consisting of the same objects regardless of the order of those objects) and ordered sets (sets consisting of the same objects in the same order).

Ninety-six first grade subjects were run, in four groups of 24 each. In one group the subjects were required to learn identity of sets for 56 trials and then equipollence for a further 56 trials. In a second group this order of presentation was reversed. In a third group the subjects learned first ordered sets and then identity and in the fourth group, order followed by identity. Our empirical aim was to establish the most adequate sequence (in terms of efficient learning) of these concepts. More specifically we were asking this kind of a question, "Is it easier for six year old children to learn to match sets, in the sense of matching the number of objects in those sets, after they have learned to match the actual objects in those sets, or vice versa?" At the same time we were specifically interested in the level of acquisition of each of those concepts in children at the beginning of the first grade. And, finally, as discussed earlier, we wished to determine whether children of this age can acquire this kind of concept when every stimulus display representing the concept is different. Our theoretical aims were as before, to examine the adequacy of the simple quantitative model described, in a concept formation situation. In this present situation of course, the test of the model was more rigorous in that any analogy to a paired associate situation was ruled out by the use of different stimulus presentations on each trial.

From our experimental results it is clear that first grade children can learn a common property, or a concept, even in a situation where every stimulus representing that concept is unique. It is true that in this situation with a severe restriction of verbal instruction the concept of equipollence proved to be quite difficult.

A particularly interesting, and unexpected empirical finding was that in no case did prior learning on one of the concepts—order, identity or equipollence—appreciably facilitate learning on a second of these concepts. This is intuitively surprising when one recollects the overlap between these concepts—for example ordered sets are always identical and identical sets are of course, always equipollent. In this experiment, if the learning task was, for example, equipollence of sets one-third of the equipollent sets were ordered and one-third were identical. Hence one would expect that prior learning on identity would have a facilitating effect on learning equipollence. Despite this, prior training on identity of sets did not increase the level of achievement or rate of learning on equipollence. Similarly neither prior training on equipollence nor on order improved learning on identity of sets.

EXPERIMENT III

In this study we compared the rate of learning in two experimental situations—one in which stimulus displays were presented individually in the usual way and the other in which the same stimulus displays were presented by means of colored slides to groups of four children. The concept to be learned was identity of sets and in both situations the children were required to respond by pressing one of two buttons, depending upon whether the stimulus display on that trial was identical or non-identical. Of the 64 subjects 32 were from first grade, 32 from kindergarten classes.

The results of this investigation suggest that presentation by slides is a less effective learning situation and the younger the child the more this finding seems to apply. At all levels of difficulty of stimulus displays the kindergarten children learned more efficiently when the stimuli were presented to them in individual sessions. On the basis of experimental evidence "difficulty" of a stimulus display is defined here in terms of the number of objects in each of the pairs of sets of objects displayed and the reader will recall that this varied from one to a maximum of three objects in a set.

With one or two element sets displayed on a trial Grade 1 subjects learned only slightly better in the individual session

situation than in the slide situation, but when the task was more difficult (stimulus displays of three element sets) the individual learning situation was clearly the most adequate. It should be emphasized that the individual session was strictly experimental so that the amount of interaction between subject and experimenter was paralleled in both individual and slide situation.

Why these two experimental situations should produce different results in terms of efficiency of learning is not yet clear to us. One possibility is the following. It has been shown, both with lower organisms [7] and with young children [8] that the ideal situation for learning is when stimulus, response and reinforcement are contiguous. In the individual session situation these requirements were met, the response buttons were 1½" below the stimulus displays, the reinforcement (colored lights, one for a correct response and one for an incorrect response) were 1" from the stimuli. In the slide presentation situation whereas the stimulus displays and reinforcements were immediately adjacent to each other, the response button which was at the child's hand, was about 3 feet from the screen onto which the stimulus display was projected. Experimentally it has been shown [8] that with children of this age group a separation of 6 inches is sufficient to interfere with efficient learning.

EXPERIMENT IV

Thirty-six kindergarten children in three groups of twelve each were run for sixty trials a day on two successive days in individual experimental sessions during which they were required to learn equipollence of sets.

On Day 1 the stimulus displays presented to the subjects on each trial differed in color between the three groups but were otherwise the same. In Group I all displays were in one color, black. In Group II equipollent sets were presented in red and non-equipollent sets in yellow. For the first twelve trials in Group III equipollent sets were presented in red and non-equipollent sets in yellow, for the remaining 48 trials on that day the two colors were gradually fused until discrimination between them was not possible. On Day 2 all sets were presented to all three Groups in one color, black.

In Group I then, we simply have a situation in which the subject has two days practice under the same conditions with the concept of equipollence. In Group II the child does not need to learn equipollence on Day I, if he simply responds to the *color* difference between equipollent and non-equipollent sets (a very simple discrimination) he will be correct on every trial. If he learns anything about equipollence of sets on the first day therefore, we assume this to be a function of incidental learning. If incidental learning is effective his performance on Day II when the color cue is dropped should at the least, be better than the performance of children in Group I on the first day. In Group III where we give the child the discriminative cue of color difference in the first trials and then very slowly withdraw that cue, the child continues to search the stimulus displays very closely for the color difference, and is thus obliged to pay very close attention to the stimuli.

Of the three groups only Group II approached perfect learning on Day I. In this Group only color discrimination was necessary. Both the other groups did not improve over the first 60 trials, although Group III showed initial improvement over those trials where the color cues remained discriminable. On the second day Group I showed no improvement and the learning curves for this group and for Group II were practically identical. In Group III on the other hand, the results were conspicuously better on the second day than were those of any other group. Apparently the task we had chosen was

very difficult—in view of the fact that no improvement was shown by Group I over a total of 120 trials—but even in this case the conditions of Group III, where the children were forced to pay very close attention to the stimuli, significantly enhanced learned.

Experiment V

In the experiments described up to this point, we have been concerned with the effect upon learning of variations in the stimulus displays, in the present experiment we introduce variations in the methods of response. Specifically three groups, each composed of 20 kindergarten children, were taken individually through a sequence of 60 trials on each of two successive days for a total of 120 trials.

In Group I the child was presented with pictures of two sets of objects and indicated, by pressing one of two buttons, whether the sets "went together" or did not "go together" (were equipollent or non-equipollent).

In Group II the child was presented with one display set and *two* "answer" sets and was required to choose the "answer" which "went together" with the display set.

In Group III the child was presented with one display set and *three* "answer" sets and made his choice from the three possible answers.

This situation has immediate reference to teaching methodology in the sense that Group II and Group III represent a multiple choice situation. In Group I where the child is required to identify either the presence of the concept or its absence on each trial, the situation is comparable to one in which the child must indicate whether an equation or statement is correct or incorrect.

On the first day each group of children learned the task as described above. On the second day they were run on an alternative method. Specifically Group I was run under Group III conditions and Groups II and III were run in the Group I situation.

Although we intend to run a further group of children in the present experimental situation before we attempt a fine analysis of these results, some clear and interesting empirical results have become apparent. In Group II—where the subjects were required to choose from one of two available responses—the subjects learned slightly more quickly and to a slightly better level of achievement on Day 1 than in the other Groups but, on the second day, when the experimental conditions were shifted, Group II subjects did considerably less well than the subjects in the other two Groups.

When we examined, separately, the data from subjects achieving criterion of 12 successive correct responses on the first day and those who did not achieve that criterion, the more successful method was clearly that used in Group I. The subjects in this Group were conspicuously more successful than the other Groups in dealing with the new situation on the second day, making in fact, no errors on that day from trial 30 to trial 60. Group III achieved perfect scores on the second day only in the last six trials, and Group II never reached that level on Day 2, although they, like the other criterion subjects, had achieved perfect learning on the first day.

It appears that the method used with Group I, where subjects were required to recognize the presence or absence of some property on each trial is the more successful method in establishing understanding of a concept well enough to permit transfer to a different situation.

If we assume, as seems reasonable, that in concept learning transfer from one situation to another is a good indication of the level of understanding of the concept learned, then a multiple choice method of response with only two available responses, seems a particularly ineffective method of acquisition. Of the three response methods

compared in this experiment, that requiring the child to identify the presence or absence of the concept to be learned on each trial was considerably the most successful.

THEORETICAL ANALYSIS

The mathematical model described at the beginning of this paper was intended for a paired associate situation. We applied it to our first experimental concept formation data (Experiment I) where six stimulus displays represented two concepts, by assuming that each of the stimulus displays involved could be treated as if it were an independent item. In effect we treated the situation as if it were a paired-associate situation with a list of six independent items to be learned. In subsequent experiments however, where every stimulus display representing a concept was different so that no stimulus display was ever repeated for any one subject, this would have been analogous to assuming a "paired associate list" with as many items in it as there were trials in the experiment.

This, therefore, required some interpretation of our situation which would enable us to define a "stimulus item" in some other way than as identified with any specific stimulus display. There are several methods of analysis which may be used in this situation, most of which we examined but we mention here only two of these.

In each case the specific stimulus to be learned is defined as some property of the stimulus display, in other words it is identical with the concept itself. Consider, for example, experimental trials where the subject must learn equipollence of sets. We may here consider all equipollent stimulus displays as representing the one concept, equipollence, and all non-equipollent sets as representing the concept of non-equipollence. So that we have two concepts to be learned, or in a paired-associate sense we have a list of two items. If we wish to do so we could, in fact, consider all stimulus displays, equipollent and non-equipollent, as representing the one concept of equipollence and analyze our data as if we had a single concept item to be learned.

The detailed analysis which we performed for Experiment I, as instanced by the statistics listed in Table I, were performed in Experiment II for two of our groups combined. In this case we had sufficient subjects (48) to make predicted and observed comparisons of some of the sequential statistics that would not otherwise, with fewer observations, have been possible. We analyzed these data from a "two item list" point of view and the results were good. In view of the wide assumptions we made in this situation that 2 abstract concepts could be treated exactly as if they were specific stimulus displays in themselves, we feel almost inclined to say that the results were surprisingly good! Again, as an example of these results, we present below in Table II some of the obtained and predicted statistics for these data.

TABLE II

PREDICTED AND OBSERVED STATISTICS FOR CONCEPT FORMATION EXPERIMENT II (N=48) WITH NON-REPEATED STIMULUS DISPLAYS

Statistic	Predicted	Observed
1. Expected errors per item per S	....	2.64
2. s.d. of errors per item per S	2.38	2.61
3. Expected errors before first success	.27	.25
4. s.d. of errors before first successes	.58	.50
5. Expected number of success runs	2.88	2.91
6. Expected number of alternations of success and failure	3.99	4.07
7. Net difference between success following error and error following error	1.55	1.66
8. Net difference between error following error and error following success	−1.36	−1.48

SOME IMPLICATIONS OF THE MODEL

We indicated earlier that certain implications are implicit in the assumptions of the quantitative theory we have used to analyze the present concept formation experiments. These implications we feel, provide a more rigorous test of the model than do the numerous statistics we have previously calculated, examples of which are given in Table I and II.

To illustrate these implications we would remind the reader that a basic assumption of the theory employed is that, over trials before the correct association is made between stimulus display and its appropriate response, the probability of a correct response is at chance level, which is to say that the subject is simply guessing. Moreover the "guessing level" is assumed to remain constant over these trials. Immediately after the correct association is made, the probability of a correct response is one. This model is in fact, one of the class known as "all-or-none" models [6] in the sense that it is assumed that learning occurs, not in an incremental fashion over trials, but immediately and completely on one trial (hence the "all-or-none" description). If therefore, we look only at the trials before the final error occurs for each subject (an event directly observable in the subject's protocol) according to our model the probability of a correct response over these trials remains constant at some "guessing" level. Hence over these trials, instead of the traditional negatively accelerated learning curve, our theory would predict a horizontal line. This prediction can be immediately tested by a simple χ^2 goodness-of-fit test for stationarity. The precise statistic employed is described in Suppes and Atkinson [10].

At the same time the reader will recollect that repeated independent trials with a fixed set of possible outcomes for each trial and with constant probabilities throughout the trials exactly defines Bernoulli trials. A test for independent trials is immediately possible—again the appropriate statistic is described in Suppes and Atkinson [10]. For the experiments with more than two responses it is simplest to apply the test in terms of the two categories of correct and incorrect responses. The probability of k successes in n Bernoulli trials has the binomial distribution

$$\binom{n}{k} p^k q^{n-k}$$

If therefore, we divide our data into blocks of, say 4, trials making the best estimate of p by using the mean probability of a correct response over all trials before the final error, our n in this case would be 4. As from 0 to 4 errors are possible in a block of 4 trials we can estimate the predicted frequencies of successes for each value of k (k=0, 1, 2, 3, 4) and compare by means of a χ^2 goodness-of-fit test with the frequencies obtained from the data. Or we may make an even more stringent test by considering the frequencies of specific sequences of successes and failures in every block of four responses. There are 16 combinations of successes and failures possible in four trials and we can predict a frequency of occurrence for each of these combinations. These predicted frequencies can again be compared with the empirical frequencies.

Not all the experimental data from the experiment mentioned in this report have yet been analyzed by the methods indicated above but the tests enumerated have been applied to the data from both groups in Experiment I and all 8 subgroups in Experiment II. In all ten cases the tests for stationarity and independence of trials were non significant so that these assumptions were well confirmed.

As further examples of some of our theoretical and empirical results we present below in Figure 1 the empirical and predicted histogram for the binomial distribution of correct responses in blocks of 4

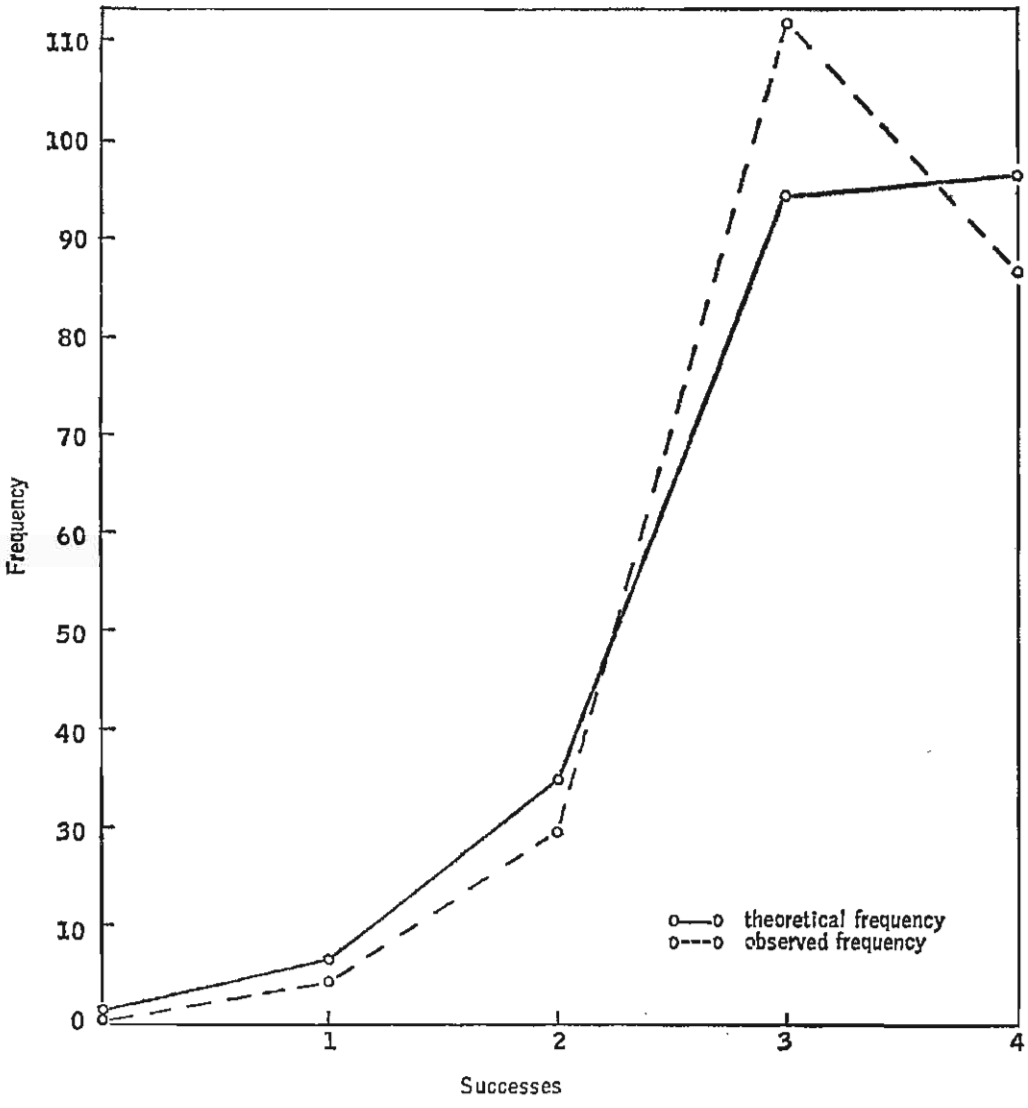

FIG. 1. Empirical and predicted histogram for binomial distribution of correct responses in blocks of 4 trials (Identity of sets experiment).

trials for the 48 subjects in Exp. 2 whose learning task was identity of sets. In Table III we present a frequency distribution for predicted and obtained number of specific sequences of errors and successes in blocks of 4 trials for the group of 24 subjects who learned equipollence of sets *after* having learned identity of sets first. These examples provide fairly typical results of this kind of analysis as far as it is completed to date, and supply quite encouraging confirmation of the present theoretical position.

SUMMARY

A series of experiments in children's concept formation has been presented and some of the empirical and theoretical results listed. Some of our findings which would seem to have immediate import to the pedagogical situation are:

1. Learning is more efficient if the child

TABLE III

FREQUENCY DISTRIBUTION OF SEQUENCES OF ERRORS AND SUCCESSES OVER BLOCKS OF 4 TRIALS (EXP. II. LEARNING TASK-EQUIPOLLENCE OF SETS *after* IDENTIFY OF SETS. N=24)

	Obtained Frequency	Predicted Frequency
0000	0	1.37
1000	1	1.40
0100	3	1.40
0010	3	1.40
0001	0	1.40
1100	4	5.37
0110	8	5.37
0011	4	5.37
1001	5	5.37
1010	2	5.37
0101	2	5.37
1110	21	20.55
1101	23	20.55
1011	16	20.55
0111	34	20.55
1111	73	78.58

who makes an error is required to make the correct response in the presence of the stimulus to be learned (Exp. I).

2. Incidental learning does not appear to be an effective method of acquisition for young children. In Exp. IV the group of children who responded to a color discrimination in a concept formation situation, did not subsequently give any indication of having learned the underlying concept.

3. A condition which focuses the child's attention upon the stimuli to be learned, enhances learning (Exp. IV).

4. Transfer of a concept is more effective if the learning situation has required the subject to recognize the presence or absence of a concept in a number of stimulus displays than if learning has involved matching from a number of possible responses. At the same time a multiple answer situation involving three responses is more effective than one involving only two possible responses (Exp. V).

5. A young child's learning tends to be very specific. In Exp. II prior training on one concept did not improve learning on a related concept.

Some of these results are unexpected. For example in Exp. IV we found incidental learning to be quite inefficient although much reliance has been placed on its effectiveness in the teaching situation, especially in the case of younger children.

Moreover, the reader will recall that at least two of our experiments achieved results not typical of adult behavior but confirming experimental results with lower organisms. We believe this to be an important point to hold in mind. At the same time the small child can, of course, be taught quite complex concepts. For example in the *Sets and Numbers* program we find that the average first grader can solve simple equations which involve the use of letters as variables, can alternate between set operations and arithmetic operations as required by the problem at hand or can deal with problems in which numbers are represented by two different kinds of notation. But the intellectual capacity of the child is one which the adult, as he himself shares it, can readily take into account. On the other hand it is difficult to appreciate that variables not generally important in adult learning behavior—such as slight variations in the response required, or the immediate overt correction of an incorrect response—may enhance or seriously interfere with learning in the young child.

In this report we have also presented a verbal description of a mathematical model of learning and in our minds of equal importance with the empirical conclusions discussed above is the success we have achieved in making detailed quantitative behavior predictions, based on this theory.

REFERENCES

1. Estes, W. K. *The Statistical Approach to Learning Theory.* In S. Koch (Ed.), *Psychology: A Study of a Science.* Vol. 2. McGraw-Hill Book Company, Inc. (1959), pp. 380–491.

2. Bower, G. H. "Application of a Model to Paired Associate Learning," *Psychometrika* (1961), 26:3, 255–280.

3. Suppes, P. *Sets and Numbers.* Stanford, California, 1961.

4. Suppes, P. and Ginsberg, R. "Application

of a Stimulus Sampling Model to Childrens Concept Formation of Binary Numbers, with and without an Overt Correction Response," *J. Exp. Psychol.* (In press.)

5. Burke, C. J., Estes, W. K. and Hellyer, S. "Rate of Verbal Conditioning in Relation to Stimulus Variability," *J. Exp. Psychol.* (1954), 48:153–161.

6. Hull, C. C. and Spence, K. W. "'Correction' vs. 'Non-Correction' Method of Trial-and-Error Learning in Rats," *J. Comp. Psychol.* (1938), 25:127–145.

7. Murphy, J. V. and Miller, R. E. "The Effect of Spatial Contiguity of Cue and Reward in the Object-Quality Learning of Rhesus Monkeys," *J. Comp. Physiol. Psychol.* (1955), 48: 221–229.

8. Murphy, J. V. and Miller, R. E. "Spatial Contiguity of Cue, Reward, and Response in Discrimination Learning by Children," *J. Exp. Psychol.* (1959), 58:485–489.

9. Suppes, P. and Ginsberg, R. "A Fundamental Property of All-or-None Models, Binomial Distribution of Responses Prior to Conditioning, with Application to Concept Formation in Children," *Psychol. Rev.* (In press.)

10. Suppes, P. and Atkinson, R. C. *Markov Learning Models for Multiperson Interactions.* Stanford University Press, 1960, p. 56.

11. *Ibid.*, p. 57.

Reprint Series of
The Stanford Institute for Mathematical Studies in the Social Sciences

1. "A Note on the Menger-Wieser Theory of Imputation," Hirofumi Uzawa, *Zeitschrift für Nationalökonomie*, Vol. XVIII, No. 3, 1958, pp. 318–34.
2. "On a Class of Capacitated Transportation Problems," Harvey M. Wagner, *Management Science*, Vol. 5, No. 3, April 1959, pp. 304–18.
3. "The Case for 'Revealed Preference'," Harvey M. Wagner, *The Review of Economic Studies*, Vol. XXVI, No. 3, June 1959, pp. 178–89.
4. "Chains of Infinite Order and Their Application to Learning Theory," John Lamperti and Patrick Suppes, *Pacific Journal of Mathematics*, Vol. 9, No. 3, 1959, pp. 739–54.
5. "Competitive Stability Under Weak Gross Substitutability: The 'Euclidean Distance' Approach," Kenneth J. Arrow and Leonid Hurwicz, *International Economic Review*, Vol. 1, No. 1, January 1960, pp. 38–49.
6. "Prices of the Factors of Production in International Trade," Hirofumi Uzawa, *Econometrica*, Vol. 27, 3, July 1959, pp. 448–68.
7. "Stability and Non-Negativity in a Walrasian Tâtonnement Process," Hukukane Nikaidô and Hirofumi Uzawa, *International Economic Review*, Vol. 1, No. 1, January 1960, pp. 50–59.
8. "The Capacity Method of Quadratic Programming," H. S. Houthakker, *Econometrica*, Vol. 28, 1, January 1960, pp. 62–87.
9. "Optimization, Decentralization, and Internal Pricing in Business Firms," Kenneth J. Arrow, Stanford University, *Contributions to Scientific Research in Management*, pp. 9–18.
10. "Classification Procedures Based on Dichotomous Response Vectors," Herbert Solomon, Stanford University, *Contributions to Probability and Statistics*, 1960, pp. 414–23. Stanford University Press.
11. "The Work of Ragnar Frisch, Econometrician," K. J. Arrow, *Econometrica*, Vol. 28, 2, April 1960, pp. 175–92.
12. "Best Linear Index Numbers of Prices and Quantities," H. Theil, *Econometrica*, Vol. 28, 2, April 1960, pp. 464–80.
13. "Walras' Tâtonnement in the Theory of Exchange," H. Uzawa, *The Review of Economic Studies*, Vol. XXVII, No. 3, pp. 182–94.
14. "Stability of the Gradient Process in n-Person Games," K. J. Arrow and Leonid Hurwicz, *J. Soc. Indust. Appl. Math.*, Vol. 8, No. 2, June 1960.
15. "Decision Theory and the Choice of a Level of Significance for the t-test," K. J. Arrow, Stanford University, *Contributions to Probability and Statistics*, 1960, pp. 70–78. Stanford University Press.
16. "Some Asymptotic Properties of Luce's Beta Learning Model," John Lamperti and Patrick Suppes, *Psychometrika*, Vol. 25, No. 3, September 1960, pp. 233–41.
17. "Some Remarks on the Equilibria of Economic Systems," K. J. Arrow and L. Hurwicz, *Econometrica*, Vol. 28, No. 3, July 1960, pp. 640–46.
18. "Group and Individual Performance in Problem Solving Related to Previous Exposure to Problem, Level of Aspiration, and Group Size," Irving Lorge and Herbert Solomon, *Behavioral Sciences*, Vol. 5, No. 1, January 1960, pp. 28–38.
19. "Some Examples of Global Instability of the Competitive Equilibrium," Herbert Scarf, *International Economic Review*, Vol. 1, No. 3, September 1960, pp. 157–73.
20. "Decentralization and Computation in Resource Allocation," Kenneth J. Arrow and Leonid Hurwicz, "Essays in Economics and Econometrics," 1960, pp. 34–104, University of North Carolina Press.

21. "Locally Most Powerful Rank Tests for Two-Sample Problems," Hirofumi Uzawa, *The Annals of Mathematical Statistics*, Vol. 31, No. 3, September 1960, pp. 685–702.

22. "Price-Quantity Adjustments in Multiple Markets with Rising Demands," Keoneth J. Arrow, Stanford University, *Mathematical Methods in the Social Sciences*, 1959, pp. 3–15, Stanford University Press.

23. "Optimality and Informational Efficiency in Resource Allocation Processes," Leonid Hurwicz, University of Minnesota, *Mathematical Methods in the Social Sciences*, 1959, pp. 27–46, Stanford University Press.

24. "Preference and Rational Choice in the Theory of Consumption," Hirofumi Uzawa, Stanford University, *Mathematical Methods in the Social Sciences*, 1959, pp. 129–48, Stanford University Press.

25. "A Stationary Inventory Model with Markovian Demand," Samuel Karlin, Stanford University, Augustus J. Fabens, Dartmouth College, *Mathematical Methods in the Social Sciences*, 1959, pp. 159–75, Stanford University Press.

26. "The Optimality of (S,s) Policies in the Dynamic Inventory Problem," Herbert Scarf, Stanford University, *Mathematical Methods in the Social Sciences*, 1959, pp. 196–202, Stanford University Press.

27. "A Random-Walk Model for Choice Behavior," W. K. Estes, Indiana University, *Mathematical Methods in the Social Sciences*, 1959, pp. 265–76, Stanford University Press.

28. "Measures of Worth in Item Analysis and Test Design," Herbert Solomon, Stanford University, *Mathematical Methods in the Social Sciences*, 1959, pp. 330–47, Stanford University Press.

29. "Stimulus-Sampling Theory for a Continuum of Responses," Patrick Suppes, Stanford University, *Mathematical Methods in the Social Sciences*, 1959, pp. 348–65, Stanford University Press.

30. "Optimal Policies for a Multi-Echelon Inventory Problem," Andrew J. Clark and Herbert Scarf, *Management Science*, Vol. 6, No. 4, July 1960, pp. 475–90.

31. "Market Mechanisms and Mathematical Programming," Hirofumi Uzawa, *Econometrica*, Vol. 28, No. 4, October 1960, pp. 872–81.

32. "On the Formation of Prices," Takashi Negishi, *International Economic Review*, Vol. 2, No. 1, January 1961, pp. 122–26.

33. "Application of Stimulus Sampling Theory to Situations Involving Social Pressure," Patrick Suppes and Franklin Krasne, *Psychological Review*, Vol. 68, No. 1, 1961, pp. 46–59

34. "Test of Stimulus Sampling Theory for a Continuum of Responses with Unimodal Noncontingent Determinate Reinforcement," Patrick Suppes and Raymond W. Frankmann, *Journal of Experimental Psychology*, Vol. 61, No. 2, 1961, pp. 122–32.

35. "A Comment on Newman's 'Complete Ordering and Revealed Preference,'" Hirofumi Uzawa, *The Review of Economic Studies*, Vol. XXVIII, No. 2, February 1961, pp. 140–41.

36. "Neutral Inventions and the Stability of Growth Equilibrium," Hirofumi Uzawa, *The Review of Economic Studies*, Vol. XXVIII, No. 2, February 1961, pp. 117–24.

37. "Additive Logarithmic Demand Functions and the Slutsky Relations," Kenneth J. Arrow, *The Review of Economic Studies*, Vol. XXVIII, No. 3, August 1961, pp. 176–81.

38. "Behavioristic Foundations of Utility," Patrick Suppes, *Econometrica*, Vol. 29, No. 2, April 1961, pp. 186–202.

39. "A Generalization of Stimulus Sampling Theory," Richard C. Atkinson, *Psychometrica*, Vol. 26, No. 3, September 1961, pp. 281–90.

40. "Capital-Labor Substitution and Economic Efficiency," K. J. Arrow, H. B. Chenery, B. S. Minhas, and R. M. Solow, *The Review of Economics and Statistics*, Vol. XLIII, No. 3, August 1961, pp. 225–50.

41. "The Observing Response in Discrimination Learning," Richard C. Atkinson, *Journal of Experimental Psychology*, Vol. 62, No. 3, 1961, pp. 253–62.

42. "The Philosophical Relevance of Decision Theory," Patrick Suppes, *Journal of Philosophy*, Vol. LVIII, No. 21, October 12, 1961, pp. 605–14.

43. "Monopolistic Competition and General Equilibrium," Takashi Negishi, *Review of Economic Studies*, Vol. 28, No. 3, 1961, pp. 196–201.
44. "Constraint Qualifications in Maximization Problems," Kenneth J. Arrow, Leonid Hurwicz, and Hirofumi Uzawa, *Naval Research Logistics Quarterly*, Vol. 8, No. 2, June 1961, pp. 175–91.
45. "Advertising Without Supply Control: Some Implications of a Study of the Advertising of Oranges," Marc Nerlove and Frederick V. Waugh, *Journal of Farm Economics*, Vol. XLIII, No. 4, Part I, November 1961, pp. 813–37.
46. "Stochastic Learning Theories for a Response Continuum with Non-Determinate Reinforcement," Patrick Suppes and Joseph L. Zinnes, *Psychometrika*, Vol. 26, No. 4, December 1961, pp. 373–90.
47. "On a Two-Sector Model of Economic Growth," by Hirofumi Uzawa, *Review of Economic Studies*, Vol. XXIX, No. 1, 1962, pp. 40–47.
48. "Test of Some Learning Models for Double Contingent Reinforcement," by Patrick Suppes and Madeleine Schlag-Rey, *Psychological Reports*, February 1962, 10, pp. 259–68.
49. "Quasi-Concave Programming," by Kenneth J. Arrow and Alain C. Enthoven, *Econometrica*, Vol. 29, No. 4, October 1961, pp. 779–800.
50. "The Stability of Dynamic Processes," by Hirofumi Uzawa, *Econometrica*, Vol. 29, No. 4, October 1961, pp. 617–31.
51. "A Quarterly Econometric Model for the United Kingdom," by Marc Nerlove, *The American Economic Review*, Vol. LII, No. 1, March 1962, pp. 154–76.
52. "Experimental Analysis of a Duopoly Situation from the Standpoint of Mathematical Learning Theory," by Patrick Suppes and J. Merrill Carlsmith, *International Economic Review*, Vol. 3, No. 1, January 1962, pp. 60–78.
53. "Experimental Studies of Mathematical Concept Formation in Young Children," by Patrick Suppes and Rose Ginsberg, *Science Education*, Vol. 46, No. 3, April 1962, pp. 230–40.