In N. J. Smelser and P. B. Baltes, (Eds.), 2001. *International Encyclopedia of the Social & Behavioral Sciences.* Oxford: Elsevier, 1026–1032.

Axiomatic Theories

Axiomatic development of theories is common practice in pure mathematics and is also now widely used in many sciences. The main ingredients of the methods for axiomatizing theories are the following: statement of the primitive concepts of the theory, statement of the prior mathematics basis assumed, statement of the axioms, characterization of models of the theory and a definition of two such models having the same structure. Theories formulated in this way can easily satisfy the standard set-theoretical approach to axiomatization. The further step of formalizing the language of the theory, especially in the case of elementary theories, can lead to specific positive and negative results about the axiomatizability of theories in restricted languages.

1. Historical Background

Of all the remarkable intellectual achievements of ancient Greek civilization, none has had greater subsequent impact than the development of the axiomatic method of analysis. No serious traces are to be found in the earlier civilizations of Babylon, China, Egypt, or India. The exact history of the beginnings is not known, but elements that can now be identified emerged in the fifth century BC. A good reference is Knorr (1975). What can be said, and is important for the subsequent discussion here, is that, already in the next century, the fourth century BC, the detailed and elaborate theory of proportion of Eudoxus emerged in quite a clear and definite axiomatic form, most of which is preserved in Book V of Euclid's *Elements* (Euclid 1925). What is important about Eudoxus's work, and even the commentaries of the work of this time, for example, by Aristotle, in the *Posterior Analytics* (1994 74a–17), is that the theorems were proved, not for single geometric objects, but for magnitudes in general, when applicable. The recognition of the correct abstraction in the general concept of magnitude, its technical and thorough implementation by Eudoxus and the philosophical commentary by Aristotle, represent a genuinely new intellectual development. The language of Eudoxus's famous Definition V in Book V of Euclid's *Elements* matches in its abstractness and difficulty the standards of modern axiomatic theories in mathematics and the sciences.

Definition 5 *Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.*

The codification of the Greek axiomatic approach in Euclid's *Elements* was a great success and remained almost unchallenged until difficulties in the details were found in the eighteenth and nineteenth centuries, as discussed later.

Various scientific examples of axiomatic theories existed already in ancient Greek times and, of course, from the Greek standpoint, they were regarded as essentially homogeneous with the axiomatic theory of geometry, no sharp distinction being made between geometry and mechanics, for instance. A good example is Archimedes' set of partial qualitative axioms for measuring weights on balances. This is undoubtedly the first partial qualitative axiomatization of conjoint measurement, a form of measurement that has received both much axiomatic attention and manifold applications in modern theories of measurement in the social sciences. (For detailed treatment of conjoint measurement, see Krantz et al. 1971. Also see *Measurement Theory: Conjoint*.)

Other early examples of axiomatic theories aimed at empirical matters can be found in the large medieval literature on qualitative axioms for the measurement of weight (Moody and Clagett 1952). Even more impressive examples are to be found in the medieval literature on physics. Some of the most subtle and interesting work is that of Nicole Oresme (1968) in the fourteenth century. (See especially his treatise, *Tractatus de Configurationibus Qualitatum et Motuum*.) What is surprising is Oresme's unexpectedly subtle approach in a geometrical framework to the phenomena of intensive qualities and motions. An example for Oresme would be that of a surface being more or less white.

Finally, in the geometric axiomatic tradition of analyzing phenomena in natural science, the two great late examples of deeply original work were the seventeenth-century treatises of Huygens' *The Pendulum Clock* (Huygens 1673/1986). And, of course, as the second example, Newton's *Principia* (Newton 1687/1946). Both Huygens and Newton formulated their axioms in the qualitative geometric style of Euclid and other Greek geometers two thousand years earlier.

1.1 *Axiomatic Geometry in the Nineteenth Century*

Without question, development in axiomatic methods in the nineteenth century was the perfection and formalization of the informal Greek methods that had dominated for so many centuries. The initial driving force behind this effort was certainly the discovery and development of non-Euclidean geometries at the beginning of the century by Bolyai, Lobachevski, and Gauss. An important development later in the century was the discovery of Pasch's (1882) axiom as a necessary addition to the Euclidean formulation. Pasch found a gap in Euclid which required a new axiom, namely, the assertion that if a line intersects one side of a triangle, it must also intersect a second side. This was just the beginning for Pasch. He created the modern conception of formal axiomatic methods, which has been the central aspect of the model for axiomatic work up until present times. The axiomatic example in geometry that had the most widespread influence was Hilbert's *Grundlagen der Geometrie*, first published in 1897 and still being circulated in later editions (Hilbert 1956).

2. *Ingredients of Standard Axiomatic Practice*

Building, especially, on the work in the later part of the nineteenth century in axiomatizing geometry, early in the twentieth century there was widespread axiomatization in mathematics and, to a lesser extent, in the empirical sciences. The main ingredients of the methods for axiomatizing theories were the following: statement of the primitive concepts of the theory, statement of the prior mathematics basis assumed, statement of the axioms, characterization of models of the theory and a definition for isomorphism of two such models, proof of a representation theorem when possible, and, finally, some analysis of invariance of the models of the theory. Before turning to an explicit discussion of these ingredients, it is worth noting that the emphasis should really be on models of the theory, which is what axiomatizing a theory makes clear. For it is models of the theory, i.e., structures which satisfy the theory, as explained in Sect. 2.3, that exhibit the nature of the theory, whether it be in geometry, economics, psychology or some other science. (For an elementary, but detailed, account of the concepts discussed in this section, see Suppes (1957/1999, Chaps. 8 and 12).)

2.1 *Primitive Concepts of a Theory*

The first point to recognize in axiomatizing a theory is that some concepts are assumed as primitive. Their properties are to be stated in the axioms, and, therefore, it is important to know how many such concepts there are and what is their general formal character, as relations, functions, etc. In Hilbert's axioms for elementary plane geometry, the five primitive concepts are those of point, line, plane, betweenness, and congruence. In contrast, in psychological theories of measurement, an ordering of the stimuli or other phenomena is almost always assumed—in this case, a weak ordering; that is, a binary relation that is transitive and connected, rather than the geometric ordering of betweenness. In addition to a primitive concept of ordering, there are other relations, for example, a primitive concept of the comparison of stimulus differences or an operation of combination for extensive measurement, as in the case of subjective probability. These measurement examples also apply to a large literature on utility and subjective probability in economics. In psychological theories of learning, even of the simplest nature, we would need such concepts as that of stimulus, response and a conditioning relation between stimulus and response. Theories without much more elaboration would require the important concept of similarity or resemblance, in order to develop a theory of the fundamental psychological phenomena of generalization, discrimination and transfer.

There are also important theories that use much simpler primitive concepts. A good example would be the theory of zero-sum, two-person games in normal form. The primitive concepts are just that of two nonempty sets X and Y that are arbitrary spaces of strategies for the two persons and a numerical function M, defined on the product space $X \times Y$. The intuitive interpretation of M is that it is the payoff or the utility function for the player whose space is X. The negative of that is the payoff for the player whose space is Y. Later a representation theorem is given for finite games of this sort.

2.2 *Axioms as Defining Theories*

As is widely recognized, axioms are what intuitively characterize a theory, whether it be of geometry, game theory or a psychological theory of measurement. From a formal standpoint, something more can be said. The essence of the matter is that to axiomatize a theory is to define a certain set-theoretical predicate. This is just as valid in the empirical sciences as in pure mathematics. The axioms, of course, are the most important part of such a definition.

Because the theory of weak orderings is used widely in both economics and psychology, as part of the theory of choice, it will be useful to give a formal definition and, therefore, the axioms for a weak ordering.

Definition 1 *Let A be a nonempty set and R a binary relation on A, i.e., let R be a subset of the product space $A \times A$. A structure (A, R) is a weak ordering if and only if the following two axioms are satisfied, for every a, b and c in A:*

Axiom 1 *R is transitive on A, i.e., if aRy and bRc then aRc.*

Axiom 2 *R is connected on A, i.e., aRb or bRa.*

There are various general methodological and folklore recommendations about the way in which axioms should be written—clarity, lack of ambiguity and simplicity are familiar. More substantive results about the form of axioms are discussed in Sect. 4 on first-order formalization. There are many definite mathematical results about the form of axioms, some of which are also discussed later.

2.3 *Independence of Axioms*

An early and important substantive recommendation is that the axioms be independent; i.e., none can be derived from the others. The question then arises from a methodological standpoint: How is independence to be established? The answer is in terms of *models* of the axioms. To make the matter still more explicit, we consider *possible realizations* of the theory. These are set-theoretical structures that have the right form. For example, in the case of weak orders, a possible realization is an ordered pair consisting of a nonempty set and a binary relation on that set. Such an arbitrary pair is called a possible realization because as a possible realization it is not a *model* of the theory unless the axioms of the theory are also satisfied. An obvious example of a model of the theory of weak ordering is the pair consisting of the set of positive integers and the relation of weak inequality $\geqslant$.

The independence of a given axiom of a theory is established by finding a possible realization of the theory in which all the axioms, except the particular one in question, are satisfied. The deductive argument to show that this yields a proof of independence is intuitively obvious and will not be made in a more formal manner here. But the essence is that if the concept were not independent, but definable, then it would necessarily hold in the model, just like the remaining axioms, and so a contradiction of its both holding and not holding in the given model would be obtained. So, by *reductio ad absurdum* it must be independent. It will be useful to consider an example or two.

To show that the axiom of connectedness for weak orders is independent of the axiom of transitivity, it is necessary to take a possible realization that is also a transitive relation. So, let $A = \{1, 2, 3\}$ and $R = \{1, 2\}$. In other words, R is just the strict numerical $<$ relation on the subset $\{1, 2\}$ of A. Then it is obvious that this relation is transitive but not connected for the number 3, which is in the set A, and does not stand in the relation R to any other element in A. To show that transitivity is independent of connectedness, that is, Axiom 1 for weak orders is independent of Axiom 2, it is sufficient to take the same set A as before, but now, the relation R is the set of the ordered pairs $\{(1, 2), (2, 3), (3, 1), (1, 1), (2, 2), (3, 3)\}$. Then it is clear that any two elements in the set A are connected by the relation R, but the relation R is not transitive, for, in order to be transitive, it must also have the pairs $(1, 3)$, $(2, 1)$ and $(3, 2)$.

The examples of independence given are trivial, but it is to be emphasized that it can often be a difficult task to establish whether or not an axiom is independent of the remaining axioms; that is, whether or not it can be derived from the remaining axioms. A very familiar mathematical example, with a long history in the twentieth century, is the proof that the axiom of choice is independent of the other standard axioms of set theory.

2.4 *Padoa's Principle for Proving Independence of Primitive Concepts*

Less familiar than proving the independence of axioms is the method of using models of a theory to prove independence of the primitive concepts of the theory. To prove that a particular primitive concept of a theory, for example, the notion of congruence in Euclidean geometry, is independent of the other primitive concepts, it is sufficient to find two models of the theory, such that the domain of both models is the same, the two models are the same for all the other primitive concepts of the theory, but the two models differ only in their realization of the concept in question. Thus, to prove congruence independence, what is required are two different models of the axiom in question, which are the same for all the other concepts, such as point, line, etc., but which have two distinct notions of congruence.

In the case of weak orders, Padoa's principle can be used in an obvious way to show that the concept of the binary relation is independent of the given set A. It suffices, for example, to use the two different orderings $\leqslant$, $\geqslant$ on the set of positive integers. On the other hand, since R is connected, the set A is definable in terms of the relation R for the special case of weak orderings by taking the union of the domain and range of the relation. But for different orderings, for example, partial orderings, which are reflexive, antisymmetric and transitive on the set A, it is easy to show that the set A is an independent concept. Just let A in one model be a proper subset in the other, and elements in the relation come only from the first set.

3. *Isomorphism of Models of a Theory*

The separation of the purely set-theoretical characterization of the possible realizations of a theory from the axioms proper, which characterize models of a theory, is significant in defining certain important concepts. For example, the notion of two models of a theory being isomorphic is often said to be axiom-free, since the definition of isomorphism of the models of a theory really depends on just the set-theoretical characterization of the possible realizations. A satisfactory general definition of isomorphism for the structures that are possible realizations of any sort of theory is difficult, if not impossible, to formulate. The usual practice is to formulate a special definition for the possible realizations of some particular theory. This is what will be done here, as already illustrated in the case of binary relations. A possible realization is a set-theoretical structure that is a nonempty set and a binary relation whose domain and range are included in that set. In this case the definition of isomorphism is as follows.

Definition 2 *A binary relation structure (A, R) is isomorphic to a binary relation structure (A', R') if, and only if, there is a function f such that*

(a) $D(f) = A$ and $R(f) = A'$

(b) f is a one–one function

(c) For a, b in A, aRb if and only if $f(a)R'f(b)$.

The definition of isomorphism for possible realizations of a theory is used to formulate a representation theorem, which has the following meaning. A certain class of structures or models of the axiomatized theory is distinguished for some intuitive or systematic reason and is shown to exemplify within isomorphism every other model of the theory. This means that, given any model of the theory, there exists in this distinguished class an isomorphic model. A good example of this can be found in the axiomatic theory of extensive measurement. Given any empirical structure that represents the data about subjective judgments of probability and satisfies the axioms of the theory, there is a numerical structure satisfying the axioms that is isomorphic to the empirical structure. Note, of course, that there is not any one single such isomorphism. Different individuals can have different empirical structures realizing their subjective probabilities, but there will be for each of them a particular numerical model of the axioms that is isomorphic to each given empirical structure.

Here is a simple and obvious, but useful, example of a representation result for two-person, zero-sum games that are finite. A game (X, Y, M), as introduced earlier, is finite just in case the sets X and Y are finite. So, with the definition of isomorphism obvious from the result now to be stated, any finite game $G = (X, Y, M)$ with $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_n)$ is isomorphic to the game $G' = (I_m, I_n, M')$, and $M'(i, j) = M(x_i, y_j)$, where the notation I_k denotes the set of positive integers $1, \ldots, k$. Therefore, any finite two-person zero-sum game may be represented by a game where X and Y are initial segments of the positive integers, even if vivid substantive descriptions of the individual strategies x_i and y_j have been given.

3.1 *Invariance Theorems for Axiomatized Theories*

In addition to having representation theorems for the models of a given axiomatized theory, it is also significant and useful to have explicit invariance theorems. The intuitive idea of invariance is most naturally explained either in terms of geometric theories or measurement theories.

Given a class of models, for example, analytical models in geometry and numerical models in measurement, with respect to which any other model of the theory can be represented, the invariance theorem is an expression of how unique is the representation in terms of such an analytic model. For example, in the case of the representation theorem for axiomatic formulations of Euclidean geometry, the invariance theorem states that any analytic model is unique only up to the group of Euclidean motions. This means that for any analytic representation of Euclidean geometry, any reflections, rotations or translations of the analytic model will produce a new analytic model, also isomorphic to the set-theoretical model satisfying the qualitative synthetic axioms of Euclidean geometry. In fact, the most general theorem widens the scope to what is often called the group of generalized Euclidean transformations, namely, those that also permit a change in scale, so that no unit of measurement is fixed.

The situation is very similar in the case of theories of measurement. Given an empirical structure and a representing numerical model isomorphic to that structure, then, in the case of intensive quantities such as cardinal utility, for example, the numerical model is unique only up to the group of affine transformations; that is, transformations that multiply the unit of measurement by a positive real number and add a constant as well. This means that in fundamental measurement of a cardinal utility there is nothing but convention in the choice of a unit of measurement or a zero value.

Classical examples of important invariant theorems in physics are that structures of classical physics are invariant up to Galilean transformations, and in special relativity, invariant up to Lorenz transformations—in both these cases, further generalizations arise by also permitting changes in the units of measurement.

4. Theories with Standard Formalization

The set-theoretical framework for axiomatizing theories just discussed is used implicitly throughout the mathematical social sciences. Most of the axiomatic work that takes place in the social sciences can be put in a straightforward way within the framework just described. On the other hand, there is a tighter framework for discussing the axiomatization of theories that leads to clarification of problems that arise in the conceptual or qualitative thinking about theories in the social sciences. The purpose of this section is to describe in an informal way this more narrowly defined framework and to give a sense of the kind of results that can be obtained. Most of the results are of a negative sort, important because they show what cannot be done by apparently very natural qualitative formulations of theories of preference, of subjective probability or related kinds of qualitative measurement, or scaling problems, especially in economics and psychology, but also in anthropology, political science, and sociology.

A language with standard formalization is a language that is given a precise formulation within first-order logic. Such a logical framework can be characterized easily in an informal way. This is the logic that assumes:

(a) one kind of variable;

(b) logical constants, mainly the sentential connectives such as *and*;

(c) a notation for the universal and existential quantifiers; and

(d) the identity symbol $=$.

A language formulated within such a logical framework is often called an elementary language. Ordinarily, three kinds of nonlogical constants occur in axiomatizing a theory in such a language—the relation symbols, also called predicates, the operation symbols and the individual constants.

The grammatical expressions of the language are divided into terms and formulas, and recursive definitions of each are given. The simplest terms are variables or individual constants. New terms are built up by combining simpler terms with operation symbols in a recursive manner. Atomic formulas consist of a single predicate and the appropriate number of terms. Compound formulas are built up from atomic formulas by means of sentential connectives and quantifiers.

Possible realizations of elementary theories, i.e., theories formulated in an elementary language, assume an especially simple form. First, there is a nonempty domain for the structure. Second, corresponding to any relation symbol of the theory, there is a corresponding relation, including sets representing predicates as one-place relations. Corresponding to any operation symbols in the theory are operations defined on the domain of the structure, and, finally, individual objects of the domain correspond to the individual nonlogical constants of the theory. It is worth noting in passing that the definition of isomorphism for such elementary structures is straightforward and simple, which is not always the case for theories formulated in more complicated set-theoretical languages.

4.1 Some Positive Results About Axiomatizability

The first positive result uses two concepts. First, the set of all the finite models of a theory is called a *finitary* class of elementary structures. Second, a theory is recursively axiomatizable when there is an algorithm for deciding whether or not any formula of the language is an axiom of the theory in question.

Theorem 1 *Any finitary class of models of an elementary theory is axiomatizable, but not necessarily recursively axiomatizable.*

The importance of this result is showing that the expressive power of elementary languages is adequate for finitary classes but not necessarily for the stating of a set of recursive axioms.

A more special positive result about finitary classes of models can also be given:

Theorem 2 *Let K be the finitary class of measurement structures with respect to an elementary language L and with respect to a numerical model $\mathscr{R}$ of L such that K includes all finite models of L homomorphically embeddable in $\mathscr{R}$. If the domain, relations, functions, and constants of $\mathscr{R}$ are definable in elementary form in terms of $(Re, \leqslant, +, \cdot, 0, 1)$ then the set of sentences of L that are satisfied in every model of K is recursively axiomatizable.*

A *universal sentence* is one that has only universal quantifiers at the beginning with the scope of a quantifier the remainder of the sentence. In practice, such sentences are written as quantifier-free statements to simplify the notation. The axioms of weak ordering given earlier are of this kind. To be explicit, the conjunction of the two given axioms is a universal sentence, and this single universal sentence is, in this form, the single axiom for a weak ordering.

Theorem 3 *(Vaught's Criterion 1954). Let L be an elementary language without function symbols. A finitary class K of measurement structures (with respect to L) is axiomatizable by a universal sentence iff K is closed under submodels and there is an integer n such that if any finite model M of L has the property that every submodel of M with no more than n elements is in K, then M is in K.*

The intuitive idea of Vaught's criterion for finitary classes of models of a theory is easy to explain. Consider again weak orderings. Because the axioms involve just three distinct variables, it is sufficient to check triples of objects in the domain of an empirical structure to determine if it is a model of the theory. Generally speaking, the number of distinct variables determines the size of the submodels that must be checked to see if universal axioms are satisfied. To have a universal axiom for a theory, or what is equivalent, a finite set of universal axioms, it is necessary that the number of distinct variables be some definite number, say n. By examining submodels involving no more than n objects, it is then sufficient to determine satisfaction of the axiom or axioms, and this is the intuitive idea of Vaught's criterion.

4.2 *Some Negative Results About Axiomatizability*

To begin with, it is useful to state as a theorem what is a corollary of Vaught's criterion. It is simply the negation of it for determining when a finitary class of models of a theory is not axiomatizable in an elementary language by a universal sentence.

Theorem 4 *Let L be an elementary language without function symbols and let K be a finitary class of measurement structures (with respect to L) closed under submodels. Then K is not axiomatizable by a universal sentence of L iff for every integer n there is a finite model M of L that has the property that every submodel with no more than n elements is in K, but M is not in K.*

It is natural to interpret the negative result stated in the general form here as showing that the complexity of relationships in finitary classes satisfying the hypotheses of the theorem is unbounded, at least unbounded when the theory must be expressed by quantifier-free formulas of elementary logic.

The first application is something rather surprising. A semi-order is a structure $(A, \succ)$ satisfying the following three axioms.

Axiom 1 It is not the case that $(a \succ a)$.

Axiom 2 If $(a \succ b \;\&\; a' \succ b')$ then $(a \succ b' \vee a' \succ b)$.

Axiom 3 If $(a \succ b \;\&\; b \succ c)$ then $(a \succ d \vee d \succ c)$.

When the set A is finite, the following numerical representation holds:

$$\text{for every } a \text{ and } b \text{ in } A,\ a \succ b$$
$$\text{if and only if } f(a) > f(b) + 1.$$

What is now surprising is a result about the indistinguishability relation for semiorders, that is, the relation $\sim$ that is the negation of the characterization of the semiorder, namely, for a and b in A

$$|f(a) - f(b)| \leqslant 1 \text{ iff } a \sim b.$$

The following theorem is due to Roberts (1969).

Theorem 5 *Let L be the elementary language whose only nonlogical symbol is the binary relational symbol* $\sim$. *Then the finitary class J of measurement structures for the indistinguishability relation* $\sim$ *is not axiomatizable in L by a universal sentence.*

The next case of a negative result applying the negation of Vaught's criterion (Theorem 3) is the proof that the qualitative theory of utility differences or the qualitative theory, more generally, of various psychometric sensations is not axiomatizable by a universal sentence, contrary, of course, to the simple theory of order. This result is due to Scott and Suppes (1958).

Consider the elementary language whose only symbol is the quaternary relation symbol 'D' with the intended numerical interpretation

$$abDcd \text{ iff } f(a) - f(b) \geqslant f(c) - f(d).$$

We then define the finitary class of measurement structures for algebraic difference as consisting of all models (A, D) such that

(a) A is a nonempty finite set;

(b) D is a quaternary relation on A; and

(c) (A, D) is isomorphic to (A', Δ), where A' is a finite set of numbers and Δ is the quaternary numerical relation such that for real numbers x, y, u and v, $xy\Delta uv$ iff $x - y \geqslant u - v$.

Theorem 6 *Let L be the elementary language whose only nonlogical symbol is the quaternary relation symbol D. The finitary class* $\mathscr{D}$ *of measurement structures for (algebraic) difference is not axiomatizable in L by a universal sentence.*

The intuitive idea of the proof can be seen from construction of a ten-element structure, all of whose substructures have a numerical representation, but which does not itself have such a representation. The idea of this construction can then be generalized for arbitrary n in order to apply Theorem 4.

Using the same ideas, Titiev (1972) extended the results of Theorem 6 to additive conjoint measurement and to multidimensional scaling with the Euclidean metric. Titiev (1980) also gives a negative proof for the city block metric for $n \leqslant 3$ dimensions. Using more sophisticated logical results, it is possible to extend most of the results just stated to not being finitely axiomatizable. This means that existential quantifiers can be introduced, but the number of axioms must be finite in character. The main results here are due to Per Lindstrom, which together with the other results mentioned in this section are presented in detail in Luce et al. (1990).

Theorem 7 *The finitary class of measurement structures for algebraic difference is not finitely axiomatizable in the elementary language whose only nonlogical symbol is the quaternary relation symbol D.*

In a way, a still stronger negative result about axiomatizability can be proved for Archimedean axioms. Here is one standard formulation of such an axiom. If $a \succcurlyeq b$ then for some n, $nb \succcurlyeq a$, where nb is the combination of n copies of b. Notice that it is necessary to introduce a quantifier for integers not just for the empirical objects of the domain. Because the exact formulation of the negative result is rather complicated, the following informal theorem is given.

Theorem 8 *For any standard elementary language used to formulate a theory of measurement there is no set of elementary formulas of the language equivalent to an Archimedean axiom for the theory.*

The best way to think about the Archimedean axiom in this context is that it is a second-order axiom and, therefore, cannot be formulated by an equivalent set of formulas, in first-order logic.

Still another way of looking at this result is that to characterize the real numbers we need some sort of second-order axiom such as Dedekind completeness, the Cauchy sequences or the least-upper bound axiom, but none of these axioms, including the Archimedean axiom, can be formulated in a first-order language whose variables take real numbers as values. Narens (1974) provides a general account of Archimedean axioms in various forms.

For deeper and more general results on axiomatizability there is much recent work that can be cited, especially concerning the definability of linear orders for classes of finite models and the problem of the complexity of the class (Stolboushkin 1992, Gurevich and Shelah 1996, Hella et al. 1997).

See also: Mathematical Models in Philosophy of Science; Measurement, Representational Theory of; Measurement Theory: Conjoint; Measurement Theory: History and Philosophy; Ordered Relational Structures

Bibliography

Aristotle 1994 *Posterior Analytics*. Clarendon Press, New York

Euclid 1925 *Euclid's Elements Book V*. [Transl. T L Heath] Cambridge University Press, Cambridge, UK [1956 Dover Publications, New York]

Gurevich Y, Shelah S 1996 On finite rigid structures. *Journal of Symbolic Logic* **61**: 549–62

Hella L, Kolaitis P G, Lousto K 1997 How to define a linear order on finite models. *Annals of Pure and Applied Logic* **87**: 241–67

Hilbert D 1956 *Grundlangen der Geometrie* 8th edn. Teubner, Stuttgart, Germany

Huygens C 1673/1986 *The Pendulum Clock, or Geometrical Demonstration of Pendula as Applied to Clocks*. trans. Rockwell R J. Iowa State University Press, Ames, IA

Knorr W R 1975 *The Evolution of the Euclidean Elements*. Reidel, Dordrecht, The Netherlands

Krantz D H, Luce R D, Suppes P, Tversksy A 1971 *Foundations of Measurement*. Academic Press, New York, Vol. I

Luce R D, Krantz D H, Suppes P, Tversky A 1990 *Foundations of Measurement Vol. III, Representation, Axiomatization and Invariance*. Academic Press, New York, CA

Moody E A, Clagett M 1952 *The Medieval Science of Weights (Scientia de Ponderibus)*. University of Wisconsin Press, Madison, WI

Narens L 1974 Measurement without Archimedean axioms. *Philosophy of Science* **41**: 374–93

Newton I 1687/1946 *Principia*. trans. Cajori F. University of California Press, Berkeley, CA

Oresme N 1968 *Tractatus de Configurationibus Qualitatum et Motuum*. trans. Clagett M. University of Wisconsin Press, Madison, WI

Pasch M 1882 *Vorlesungen über Neuere Geometrie*. Verlang von Julius Springer, Leipzig, Germany

Roberts F S 1969 Indifference graphs. In: Harary F (ed.) *Proof Techniques in Graph Theory*. Academic Press, New York

Suppes P 1957/1999 *Introduction to Logic*. Van Nostrand, New York

Scott D, Suppes P 1958 Foundational aspects of theories of measurement. *Journal of Symbolic Logic* **23**: 113–28

Stolboushkin A P 1992 Axiomatizable classes of finite models and definability of linear order. In: *Proceedings of the 7th Annual IEEE Symposium on Logic in Computer Science*. IEEE Computer Society Press, pp. 64–70

Titiev R J 1972 Measurement structures in classes that are not universally axiomatizable. *Journal of Mathematical Psychology* **9**: 200–5

Titiev R J 1980 Computer-assisted results about universal axiomatizability and the three-dimensional city-clock metric. *Journal of Mathematical Psychology* **22**: 209–17

P. Suppes